Poker

Poker is an ideal setting to study probabilities. Computing the probabilities of different hands will require a variety of approaches. We will not concern ourselves with betting strategies, however. Our perspective will be solely a combinatorial one. A standard deck of playing cards with no jokers will be used. Each standard deck contains four **suits**: clubs $\clubsuit$, diamonds $\diamondsuit$, hearts $\heartsuit$ and spades $\spadesuit$. Each deck also contains thirteen different ordered **ranks**: 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace. The Ace can be considered a 1 if so desired in the ordering of the ranks. There is exactly one card of each rank and suit for a total of $13 * 4 = 52$ cards. There exist a variety of different hands in poker. In this section, only five card hands will be considered. Variations on the game of poker will be left for the homework problems. Table 1.5.1 describes the types of poker hands and ranks them in order, i.e., a hand higher up in the table beats all the hands below it in the table.

hands	**description**	**example**
royal flush	10, Jack, Queen, King, Ace of the same suit	$10\clubsuit$, Jack$\clubsuit$,Queen$\clubsuit$,King$\clubsuit$, Ace$\clubsuit$
straight flush	Five consecutive ranks of the same suit that is not a royal flush	$5\heartsuit, 6\heartsuit, 7\heartsuit, 8\heartsuit, 9\heartsuit$
four of a kind	Four cards of one rank and one of another rank	$7\heartsuit, 7\spadesuit, 7\diamondsuit, 7\clubsuit$, Jack$\diamondsuit$
full house	Three cards of one rank and two cards of another rank	$2\diamondsuit, 2\spadesuit, 2\clubsuit, 5\spadesuit, 5\diamondsuit$
flush	Five cards of the same suit that are not all consecutive ranks	$2\spadesuit, 3\spadesuit, 5\spadesuit, 10\spadesuit$, Queen$\spadesuit$
straight	Five consecutive ranks that are not all of the same suit	Ace$\spadesuit$, $2\spadesuit, 3\heartsuit, 4\clubsuit, 5\heartsuit$
three of a kind	Three cards of one rank and two cards of a second and third rank	$3\diamondsuit, 3\spadesuit, 3\clubsuit, 5\spadesuit$, Ten$\diamondsuit$
two pair	Two cards of one rank, two cards of a second rank and one card of a third rank	$3\spadesuit, 3\clubsuit$, Jack$\heartsuit$, Jack$\clubsuit$, Queen$\heartsuit$
one pair	Two cards of one rank and three cards whose ranks differ from each other	$7\spadesuit, 7\clubsuit, 2\heartsuit, 10\clubsuit$, King$\heartsuit$
other	None of the above	$3\spadesuit, 5\clubsuit, 8\heartsuit$, Jack$\clubsuit$, Queen$\heartsuit$

Using a single deck of cards and discarding the jokers, there are $\binom{52}{5} = \dfrac{52!}{5! * 47!} = \dfrac{52 * 51 * 50 * 49 * 48}{5 * 4 * 3 * 2 * 1} = 2,598,960$ different five card hands possible. The next step is to count the number of different ways each hand can possibly occur. The chart below indicates the winning order of the various hands from best to worst.

How many different poker hands of *four of a kind* exist? A *four of a kind* consists of one rank represented by all four suits and one other card. The first task is to select the rank in which to get all four suits. The second task is to select all four suits of that rank. The third and final task is to pick the fifth card. This can be done in exactly $13 * \binom{4}{4} * 48 = 624$ ways and the probability of drawing this hand is $\frac{625}{2598960} = 0.00024$ (see Table 1.5.2)

hands	**number of ways**	**probability**
royal flush		
straight flush		
four of a kind	624	0. 00024
full house		
flush		
straight		
three of a kind		
two pair		
one pair		
other		

Next, consider the three different *flush* hands. The *royal flush* is a 10, Jack, Queen, King, Ace of the same suit. It is plain to see that there are only four such hands. A *straight flush* is a *straight* of the same suit that is not a *royal flush*. First, count the number of *straights*. There exist ten different *straights*.

Ace, 2, 3, 4, 5
2, 3, 4, 5, 6
3, 4, 5, 6, 7
4, 5, 6, 7, 8
5, 6, 7, 8, 9
6, 7, 8, 9, 10
7, 8, 9, 10, Jack
8, 9, 10, Jack, Queen
9, 10, Jack, Queen, King
10, Jack, Queen, King, Ace

After selecting one of the ten *straight*s, a suit must then be selected. Hence, there exist $10 * 4 = 40$ *straights* of the same suit. However, four of those would be considered a *royal flush*. No sane poker player would hold a *royal flush* and call it a *straight flush*. The four hands that are a *royal flush* must now be subtracted from the total and there are $40 - 4 = 36$ different *straight flush* hands in poker. This concept will be needed again. It is crucial that when counting a particular hand that a better hand is not included while enumerating the lesser hand.

A *flush* consists of five cards of the same suit. There are thirteen cards of each suit and

$$\binom{13}{5} = \frac{13!}{5! * 8!} = \frac{13 * 12 * 11 * 10 * 9}{5 * 4 * 3 * 2 * 1} = 13 * 11 * 9 = 1,287$$

ways to pick five cards of any one suit. However, previous work has shown that nine of those hands are actually a *straight flush* and one of them is a *royal flush*. Hence, there are $1287 - (9 + 1) = 1,277$ hands that are a *flush* for any given suit. Next, multiply by four to select the suit for the *flush* and there are 5,108 different *flush* hands. Three more hands have their information entered in Table 1.5.3.

hands	**number of ways**	**probability**
royal flush	4	0. 00000154
straight flush	36	0. 000014
four of a kind	624	0. 00024
full house		
flush	5,108	0. 0019654
straight		
three of a kind		
two pair		
one pair		
other		

The remaining information will be computed in the homework section. When counting the number of different five card hands of a particular type, it is very important that you not improve on the type of hand that you are enumerating.

Homework

1. Complete the table of probabilities for the remaining poker hands.

2. Does the ordering of the hands in poker make sense according to the probabilities associated with each hand? Explain.

3. Shot in the back while playing poker, Wild Bill Hickok's final hand was a pair of aces and a pair of eights, now known as the **dead man's hand**. How many different dead man's hands exist?

4. What is the probability that a five card hand will contain no face cards?

5. What is the probability that a five card hand will contain at least one of each suit?

6. How many five card hands contain at least one of each of the three face cards (King, Queen, Jack)?

7. If a deck of cards contains the two jokers (one red, one black) that can be any desired card, what is the probability of *four of a kind*?

8. If a deck of cards contains the two jokers that can be any desired card, what is the probability of a *full house*?

9. If a deck of cards contains the two jokers that can be any desired card, which hand should win: a *five of a kind* or a *royal flush*? Explain your reasoning.

10. A player holds the five cards 2♣, 3♣, 6♣, 10♠ and Ace ♣ in his hand. If he discards the 10♠, (without returning it to the deck) and draws one more card, what is the probability that the result will be a *flush*?

11. A player holds the five cards 5♣, 5♢, King♢, King♠ and Ace ♣ in her hand. If she discards the Ace ♣ (without replacing it into the deck) and draws an additional card, what is the probability that the result will be *a full house*?

12. A player holds the five cards 2♣, 5♢, King♢, King♠ and Ace ♣ in her hand. If she discards the 2♣, 5♢ and Ace ♣ (without replacing them into the deck) and draws three more cards, what is the probability that the result will be

    i. a *three of a kind*;
    ii. a *three of a kind* or better?

13. In **seven card stud**, a player is dealt seven cards and then picks the best five card hand from those seven. Compute the probabilities for each of the following hands in seven card stud.

    i. *royal flush*;
    ii. *straight flush*;
    iii. *four of a kind*.

In the variation of poker known as ***Texas Hold'em****, each player is dealt two personal cards face down (and betting ensues). Three community cards are dealt face up (and more betting ensues). Finally two more community cards are dealt face up, one at a time. The object is to construct the beast five card hand using any combination of a players personal cards and the community cards (different players may use the same community cards).*

14. A player holds $A\heartsuit$ and $5\spadesuit$ as his personal cards and the three community cards are $5\clubsuit$, $3\clubsuit$ and $K\diamondsuit$. Compute the probability of each of the following hands once the final two community cards are dealt.

i. flush;
ii. straight;
iii. four of a kind;
iv. full house;
v. three of a kind.

15. John holds $K\diamondsuit$ and $K\spadesuit$ as his personal cards and the five community cards are $3\diamondsuit, 7\diamondsuit J\clubsuit$, $K\clubsuit$ and $A\diamondsuit$. If John has only one opponent, what is the probability that John loses?

16. When selecting three cards in an ordered sequence, what is the probability that the rank of the first card is strictly smaller than the rank of the second card which, in turn, is strictly smaller than the rank of the third card?