3.9 Prüfer Sequences

In Section 3.8, Cayley's Tree Formula gave the number of unequal labeled trees on n vertices as $t_n = n^{n-2}$. The classic proof of this formula is constructive in nature. It involves a problem which, at first glance, appears to be unrelated to the problem of counting labeled trees. The problem counts the number of sequences of length $n-2$ with entries from $\{1, 2, 3, ..., n\}$ where repetition is allowed. An easy application of the multiplication rule finds that there are n^{n-2} such sequences. These sequences are called **Prüfer Sequences**. The beauty of the proof is the discovery of a one-to-one and onto function between all unequal labeled trees on n vertices and the **Prüfer** sequences. As noted in Section 2.1, if a one-to-one and onto correspondence exists between domain and range sets then $|D| = |R|$. Two algorithms are presented that demonstrate such a correspondence between these two sets. These algorithms are due to Prüfer and are collectively referred to as Prüfer's proof of Cayley's Tree Formula.

The first algorithm builds a Prüfer sequence from any labeled tree on n vertices. The sequence will be built by deconstructing the tree one leaf at a time.

Algorithm tree_to_sequence (T, n)

Input: A labeled tree T with n vertices.
Output: The Prüfer sequence for tree T.

Step One: Find the leaf l in T with the smallest label.

Step Two: Record the label of the vertex adjacent to l.

Step Three: If $n-2$ labels have been created then stop. Else goto Step One where T is now the tree created by removing l and its incident edge from T.

The tree in Figure 3.9.1 will be used to demonstrate the algorithm. First identify the leaf with the smallest label. This leaf is vertex 3 and the label of the vertex adjacent to 3 (and not 3 itself) is now recorded. The label 2 is recorded giving the sequence (2) and vertex 3 is deleted. Since five labels have not been recorded the algorithm is applied to the new tree created by the deletion of vertex 3.

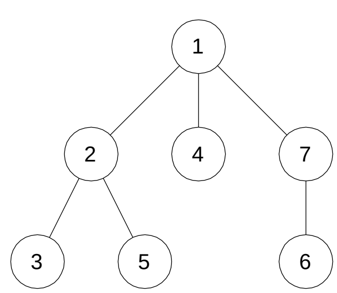


Figure 3.9.1

The smallest labeled leaf in the resulting tree (Figure 3.9.2) is 4. Thus, the adjacent label 1 is recorded giving the sequence (2,1) and vertex 4 is deleted. Five labels still have not been recorded and the algorithm is applied again to the new tree (the second tree in Figure 3.9.2). The complete process to a sequence of five labels is illustrated in Figure 3.9.2.

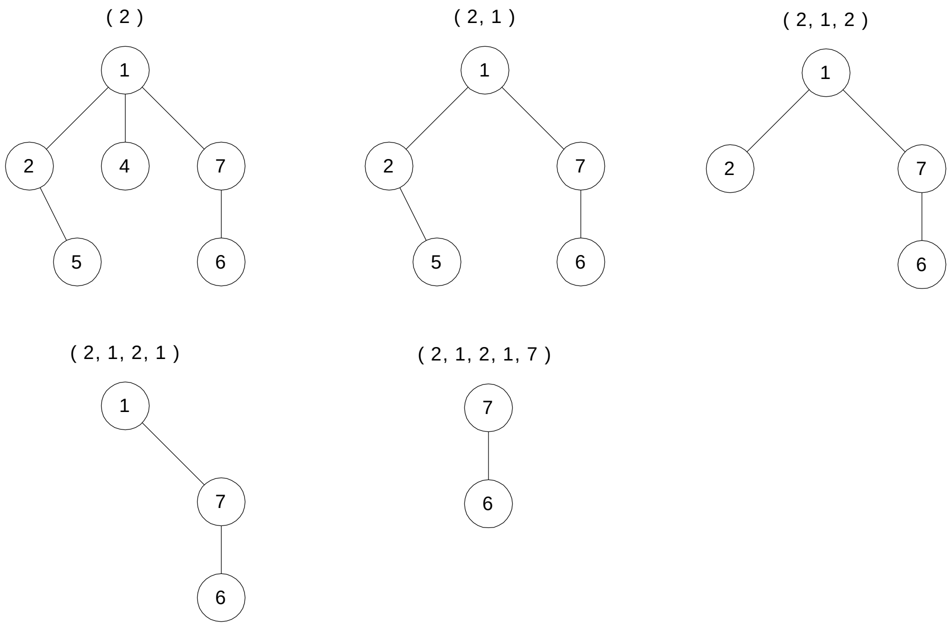


Figure 3.9.2

Thus the Prüfer sequence for the tree T of Figure 3.9.1 is $(2,1,2,1,7)$. All seven labels are not represented in this sequence and some labels are recorded more than once. Which labels are missing? The leaves. In fact, the careful reader will note that the degree of a vertex in T is one more than number of times the label is recorded in the sequence.

Reversing the process requires some bookkeeping. This is due to the fact that leaves are defined by the absence of a label in the Prufer sequence. The first step in the reverse process is to add two to the length of the Prufer sequence to determine the number of vertices in the tree. The second step is to create a list L of leaves by determining which of the n labels are not present in the Prufer sequence and, hence, are leaves.

Algorithm sequence_to_tree(S)

Input: A Prüfer sequence S.
Output: The tree T corresponding to S.

Step One: Let n be the length of the sequence plus two.

Step Two: Let $L = \{i | 1 \le i \le n \text{ and } i \notin S\}$.

Step Three: Select the smallest label j in L and let k be the first element in the sequence S. Record the $j-k$ edge. Delete j from L. If k is the only element in S then L will now contain only a single label m. Record the $k-m$ edge and stop. If k is not the only element in S then goto Step Four.

Step Four: Delete k from S. If k occurs elsewhere in S then goto Step Five. If k does not occur elsewhere in S then add k to L and goto Step Five.

Step Five: Goto Step Three with the shortened sequence S and possibly larger set L.

Consider the sequence $S=(2,1,2,1,7)$. Then $n=5+2=7$ and $L=\{3,4,5,6\}$. In L, 3 is the smallest label and 2 is the first label in S. The first edge in the tree T is the $2-3$ edge. Deleting the label 3 from L, creates the new set $L=\{4,5,6\}$. The label 2 is not the last label in S and is deleted to form $S=(1,2,1,7)$. The label 2 does occur again in S and is not added to the list L. The algorithm is now repeatedly applied until the last label in S is used. The complete process for the sequence $S=(2,1,2,1,7)$ and creation of the original tree of Figure 3.9.1 is detailed in Figure 3.9.3.

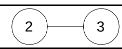
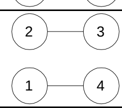
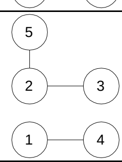
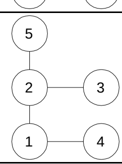
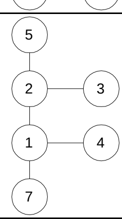
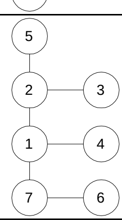
S	***L***	***T***	***L***	**Is *k* last?**	***S***	**Is $k \in S$?**	***L?***
$(2,1,2,1,7)$	$\{3,4,5,6\}$		$\{4,5,6\}$	No	$(1,2,1,7)$	Yes	$\{4,5,6\}$
$(1,2,1,7)$	$\{4,5,6\}$		$\{5,6\}$	No	$(2,1,7)$	Yes	$\{5,6\}$
$(2,1,7)$	$\{5,6\}$		$\{6\}$	No	$(1,7)$	No	$\{2,6\}$
$(1,7)$	$\{2,6\}$		$\{6\}$	No	(7)	No	$\{1,6\}$
(7)	$\{1,6\}$		$\{6\}$	Yes			
							

Figure 3.9.3

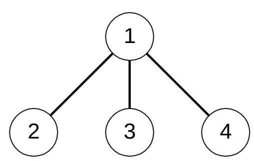
Homework

1. Construct the labeled tree from the following sequences.
   i. $(3, 1, 2, 5, 4)$
   ii. $(1, 1, 1, 1, 1)$
   iii. $(1, 2, 1, 3, 1, 4, 1, 5)$
   iv. $(4, 3, 1, 5, 4, 8)$

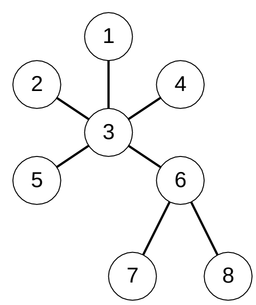
2. Construct the labeled tree from the following sequences.
   i. $(3, 3, 2, 2)$
   ii. $(6, 1, 6, 1)$
   iii. $(1, 6, 1, 6)$
   iv. $(3, 2, 5, 3, 6, 5, 1, 3)$

3. Construct the Prüfer sequence from each of the following labeled trees.

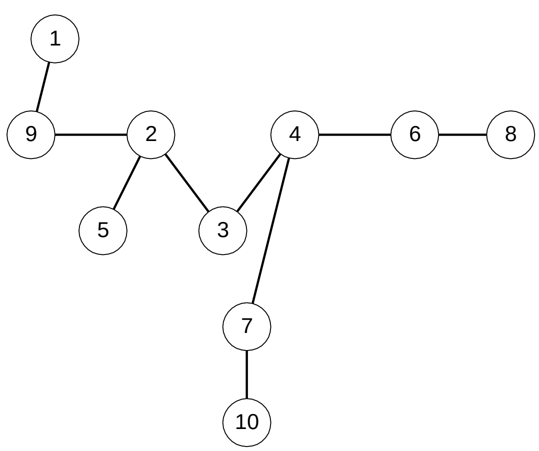
A



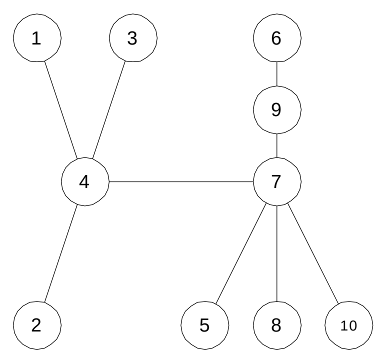
B



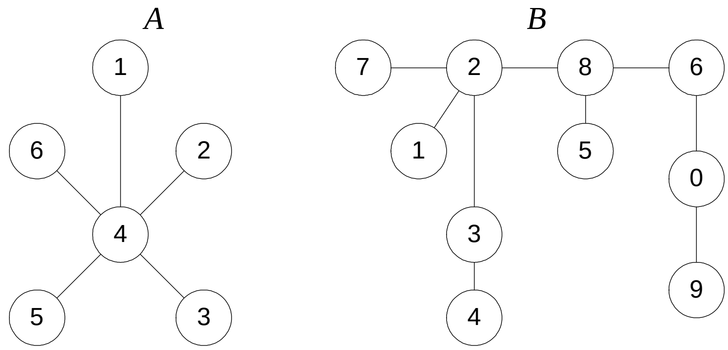
C

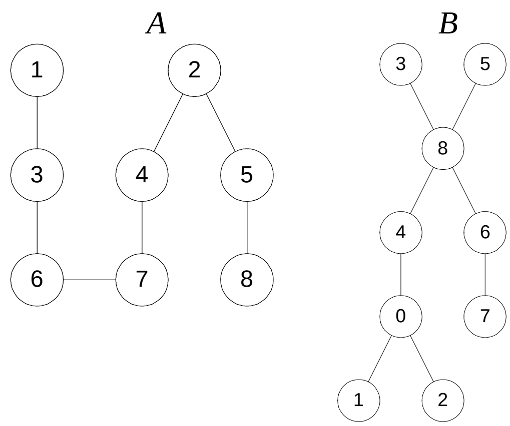


D



4. Construct the Prüfer sequence from each of the following labeled trees.





5. Use a Prüfer sequence to show there does not exist a tree of nine vertices with two vertices of degree five.

6. Use a Prüfer sequence to show there does not exist a tree of ten vertices with three vertices of degree four.

7. Describe the tree generated by a Prüfer code of 25 entries of 7.

8. Describe the tree generated by a Prüfer code with no repeated digits.

9. How many vertices exist in a tree with three vertices of degree three and all other vertices are leaves?

10. How many vertices exist in a tree with two vertices of degree four, two vertices of degree two and all other vertices are leaves?

11. Find the number of different labeled trees that exist such that $\deg(1) = \deg(2) = 3$ and all other vertices are leaves.

12. Find the number of different labeled trees that exist such that $\deg(1) = \deg(2) = \deg(5) = 4$ and all other vertices are leaves.

13. How many different labeled trees with six vertices exist such that the degree of every vertex is one or three?