5.4 Stirling Numbers

When counting various types of functions from 2.1, we quickly discovered that enumerating the number of onto functions was a difficult problem. For a domain of five elements and a range of four elements, we quickly noted that an onto function must map two elements of the domain onto a single element of the range. The rest of the mapping was a 1-1 function of the remaing three domain elements to the remaining three range elements. Thus, there are $\binom{5}{2} * 4! = 240$ different onto functions from a domain of size five to a range of size four. For a domain of six elements and a range of four elements, we note that two cases are possible when constructing an onto function. Either three elements of the domain map to a single element of the range or two pairs of elements of the domain are mapped to two separate eleements of the ranges. The number of ways to do this is left as an exercise for the reader. Most important to note is that as the difference of the sizes of the domain and ranges grows, so does the number of different ways to construct an onto function. The heart of the problem is determining the numbers of ways the domain can be partitioned into a number of non-empty subsets equal to the size of the range. This number is known as the Stirling number of the second kind.

There are two types of Stirling numbers. We will initially examine Stirling numbers of the second kind due to the fact that they occur more frequently.

The **Stirling number of the second kind**, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ or $S(n,k)$, is the number of partitions of an n-element set into k non-empty subsets. The symbol, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$, is generally read as '$n$ subset k.' For example, $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 7$ because there exist seven different ways to partition a set $S = \{1,2,3,4\}$ into two non-empty subsets as listed in Table 5.4.1.

$\{\{1,2,3\},\{4\}\}$	$\{\{1,2,4\},\{3\}\}$	$\{\{1,3,4\},\{2\}\}$	$\{\{1\},\{2,3,4\}\}$
$\{\{1,2\},\{3,4\}\}$	$\{\{1,3\},\{2,4\}\}$	$\{\{1,4\},\{2,3\}\}$	

Table 5.4.1

The Stirling number of the second kind $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} = 6$. The six different partitions are given below in Table 5.4.2.

$\{\{1,2\},\{3\},\{4\}\}$	$\{\{1,3\},\{2\},\{4\}\}$	$\{\{1,4\},\{2\},\{3\}\}$
$\{\{1\},\{2,3\},\{4\}\}$	$\{\{1\},\{2,4\},\{3\}\}$	$\{\{1\},\{2\},\{3,4\}\}$

Table 5.4.2

Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 2\}$. How many different onto functions $f : A \to B$ exist? For the function f to be onto B we must partition A into two different non-empty subsets. This can be done in $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 7$ different ways. Next we must assign each of those subsets a unique image in B. This can be done in $2! = 2$ ways. Thus, there are $7 * 2 = 14$ different possible onto functions $f : A \to B$. For $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3\}$, how many different onto functions $f : A \to B$ exist? This would be $\left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} * 3! = 36$ different onto functions $f : A \to B$.

The values of four particular pairs of parameters are simple to determine which is not true in the general case. For all positive integers n, $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = 0$ as it is impossible to partition an n-element set into no subsets. Likewise, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = 0$ for all $k > n$. For all positive integers n, $\left\{ \begin{matrix} n \\ 1 \end{matrix} \right\} = 1$ due to the fact that the only partition of a set into one non-empty subset is the set itself. Similarly, for all positive integers n, $\left\{ \begin{matrix} n \\ n \end{matrix} \right\} = 1$. The only partition of an n-element set into n subsets is the collection of all single element subsets of the original set.

The value $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\}$ is also easy to compute. The only partition of an n-element set into $n-1$ non-empty subsets will consist of one subset of size two and all other subsets will be single element subsets. Hence, selection of the unordered pair determines the partition and $\left\{ \begin{matrix} n \\ n-1 \end{matrix} \right\} = \binom{n}{2}$.

Next consider $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\}$. In order to partition an n-element set into two non-empty subsets, we need just to select a proper, non-empty subset from our n-element set. The partition will consist of this subset and its complement. It is obvious why this initial subset must not be empty. It must also be proper, otherwise its complement would be empty. There are $2^n - 2$ ways to select this subset. But, we have counted each partition twice. Selecting the subset A generates the partition $\{A, \overline{A}\}$. However selecting $\overline{A}$ also generates the same partition. Thus, $\left\{ \begin{matrix} n \\ 2 \end{matrix} \right\} = \frac{2^n - 2}{2} = 2^{n-1} - 1$.

Theorem 5.4.1: The general formula for computing the Stirling number of the second kind is $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$.

With the formula in Theorem 5.4.1 it is possible to determine the Stirling number of the second kind $S(n,k)$ without generating all possible partitions. For example,
$\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = \frac{1}{2!}\sum_{i=0}^{2}(-1)^i\binom{2}{i}(2-i)^4 = \frac{1}{2}\left[(-1)^0\binom{2}{0}2^4+(-1)^1\binom{2}{1}1^4+(-1)^2\binom{2}{2}0^4\right]$
$= \frac{1}{2}[16-2+0] = 7.$

Next let $A = \{1,2,3,4,5,6,7\}$ and $B = \{1,2,3,4\}$. How many different onto functions $f: A \to B$ exist? For the function f to be onto B we must partition A into four different non-empty subsets. This can be done in $\left\{ \begin{matrix} 7 \\ 4 \end{matrix} \right\} = 350$ different ways. Next we must assign each of those subsets a unique image in B. This can be done in $4! = 24$ ways. Thus, there are $350 * 24 = 8,400$ different possible onto functions $f: A \to B$.

Suppose a group of twelve college students needs to head to three different locations in order to gather items for a scavenger hunt. They decide to split into three different groups in order to maximize efficiency. How many different ways can they split into three different groups? Clearly, we wish to partition the twelve students into three non-empty subsets. This can be done in $\left\{ \begin{matrix} 12 \\ 3 \end{matrix} \right\} = 86,526$ different ways. If we wish to also decide the location that each group will visit we need to compute the number of different ways to select a group for each location. There are 3! different possible such assignments. Thus, the number of ways to partition the groups and decide their destinations is $\left\{ \begin{matrix} 12 \\ 3 \end{matrix} \right\} * 3! = 519,156.$

As with binomial coefficients, there exists a simple recursive formula for Stirling numbers of the second kind.

Theorem 5.4.2: For integers $n \geq k$, $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k * \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$. The careful reader will notice that this equality and subsequent proof are both very similar to the identity $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ and its combinatorial proof of Theorem 2.3.2.

Proof: Let $A = \{1,2,...,n\}$. On one hand, the number partitions of A into k non-empty subsets is $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$. On the other hand, let A_1 be the collection of all partitions $\mathcal{B}$ of A into k non-empty subsets such that $\{1\} \notin \mathcal{B}$ and let A_2 be the collection of all partitions $\mathcal{B}$ of A into k non-empty subsets such that $\{1\} \in \mathcal{B}$. Clearly $A_1 \cap A_2 = \emptyset$. Every partition of A either contains the single element set $\{1\}$ or it does not. Hence, every partition of A is contained in either A_1 or A_2. By the sum rule, the number of partitions of A into k non-empty subsets is $|A_1| + |A_2|$.

For A_2, the subset $\{1\}$ must be included in the partition. Thus, we must partition $\{2, 3, 4..., n\}$ into $k-1$ non-empty subsets and $|A_2| = \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$.

For A_1, we must partition A without including the single element subset $\{1\}$. This forces the element 1 to be part of a larger subset and there must be k non-empty subsets without regard to the element 1. There are $\left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ ways to partition $\{2, 3, 4..., n\}$ into k non-empty subsets. In order to have a partition of A into k non-empty subsets all we must do is select one of those k subsets and union it with the element 1. This can be done in k different ways. Hence, $|A_1| = k * \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\}$ and $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k * \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$. ❑

Using this formula, we can now more easily compute $S(4, 2)$. Using the Theorem 5.4.2 and some basic properties of Stirling numbers of the second kind, $\left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} = 2 * \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} + \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} = 2 * (2^{3-1} - 1) + 1 = 6 + 1 = 7.$

Stirling numbers of the second kind contribute to the enumeration of onto functions. Stirling numbers of the first kind also assist in counting a particular class of functions, namely permutations of a special form. When $|D| = |R| = n$, there exist exactly $n!$ different permutations from D to R. However, we now wish to examine the permutations and partition them according to a representation as a union of disjoint cycles.

A common way to write a permutation of n objects is in cycle form. The permutation 53142 or $\left[\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 1 & 4 & 2 \end{matrix} \right]$ represents the function f where $f(1) = 5$, $f(2) = 3$, $f(3) = 1$, $f(4) = 4$ and $f(5) = 2$. A shorter way to define f is to represent the cycles of the function. The function f is represented by the cycles $(1523)(4)$ because 1 is mapped to 5 which in turn is mapped to 2 which is mapped to 3 which maps back to 1. This is a cycle of length four. The point 4 is fixed and consists of a cycle of length 1. Graphically, the cycles of a permutation can be represented by a directed graph. The permutation $(1523)(4)$ is represented by the directed graph with two components in Figure 5.4.1. This is a case where loops must be allowed in order to represent a fixed point of a permutation.

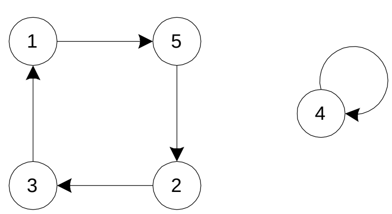

Figure 5.4.1

The **Stirling number of the first kind,** $\left[\begin{matrix} n \\ k \end{matrix} \right]$ or $s(n,k)$, is the number of permutations of n elements defined by k disjoint cycles. The symbol, $\left[\begin{matrix} n \\ k \end{matrix} \right]$, is generally read as '$n$ cycle k.' For example, $\left[\begin{matrix} 4 \\ 2 \end{matrix} \right] = 11$ because there are eleven different ways to write the permutations of $S = \{1,2,3,4\}$ into two disjoint cycles.

(1)(234)	(124)(3)	(12)(34)
(1)(243)	(142)(3)	(13)(24)
(134)(2)	(123)(4)	(14)(23)
(143)(2)	(132)(4)	

Table 5.4.3

As with $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ there are some simple formulae for some values of $\left[\begin{matrix} n \\ k \end{matrix} \right]$. As with $\left\{ \begin{matrix} n \\ 0 \end{matrix} \right\}$, $\left[\begin{matrix} n \\ 0 \end{matrix} \right] = 0$ because a permutation cannot be written without at least one cycle. For $k > n$, $\left[\begin{matrix} n \\ k \end{matrix} \right] = 0$ because a unique simple form is used for cycles and does not allow repetition of points. If repetition of points is allowed in cycle notation then a unique representation of a permutation does not exist. There is only one permutation of n elements into n cycles, namely the identity function that fixes every point and $\left[\begin{matrix} n \\ n \end{matrix} \right] = 1$. The number of permutations of n elements into one cycle is just a variation of the traveling salesman problem and $\left[\begin{matrix} n \\ 1 \end{matrix} \right] = (n-1)!$.

The value $\left[\begin{matrix} n \\ n-1 \end{matrix} \right]$ is also easy to compute. The only permutation of n points into $n-1$ cycles will consist of one cycle of length two and all other cycles will be fixed points. Note that the cycle (ab) is the same as the cycle (ba). Selection of the unordered pair determines the permutation and $\left[\begin{matrix} n \\ n-1 \end{matrix} \right] = \binom{n}{2}$.

Much like $\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k * \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}$, there is a recursive formula for Stirling numbers of the first kind.

Theorem 5.4.3: $\begin{bmatrix} n \\ k \end{bmatrix} = (n-1) * \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$. The proof of this identity is left for the homework.

Homework

1. Compute the following Stirling numbers of the second kind.

   i. $\left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\}$ ii. $\left\{ \begin{matrix} 6 \\ 4 \end{matrix} \right\}$ iii. $\left\{ \begin{matrix} 10 \\ 9 \end{matrix} \right\}$

   iv. $\left\{ \begin{matrix} 12 \\ 2 \end{matrix} \right\}$ v. $\left\{ \begin{matrix} 7 \\ 5 \end{matrix} \right\}$ vi. $\left\{ \begin{matrix} 15 \\ 3 \end{matrix} \right\}$

2. Compute the following Stirling numbers of the second kind.

   i. $\left\{ \begin{matrix} 6 \\ 3 \end{matrix} \right\}$ ii. $\left\{ \begin{matrix} 9 \\ 5 \end{matrix} \right\}$ iii. $\left\{ \begin{matrix} 1001 \\ 1000 \end{matrix} \right\}$

   iv. $\left\{ \begin{matrix} 35 \\ 2 \end{matrix} \right\}$ v. $\left\{ \begin{matrix} 23 \\ 21 \end{matrix} \right\}$ vi. $\left\{ \begin{matrix} 18 \\ 4 \end{matrix} \right\}$

3. List all the partitions of $\{1, 2, 3, 4, 5\}$ into two non-empty subsets.

4. List all the partitions of $\{1, 2, 3, 4, 5\}$ into four non-empty subsets.

5. Let $A = \{a, b, c, d, e, f, g, h\}$ and $\alpha = \{1, 2, 3, 4, 5\}$. How many functions $f : A \to \alpha$ exist? How many onto functions $f : A \to \alpha$ exist?

6. Let $A = \{a, b, c, d, e, f, g, h, i\}$ and $\alpha = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$. How many functions $f : A \to \alpha$ exist? How many onto functions $f : A \to \alpha$ exist?

7. Let $|D| = n$ and $|R| = n - 1$ where n is a finite integer. How many onto functions $f : D \to R$ exist.

8. Let $|D| = n$ and $|R| = 2$ where n is a finite integer. How many onto functions $f : D \to R$ exist.

9. A project engineer has four systems to maintain and six engineers in his group to assign to the different systems. How many different ways can the project engineer assign employees to the systems if

   i. each system must be maintained by at least one employee?

ii. each system must be maintained by at least one employee and each employee works on exactly one system?

10.

8. Prove $\left\{ \begin{matrix} n \\ n-2 \end{matrix} \right\} = \binom{n}{3} + \frac{\binom{n}{2}\binom{n-2}{2}}{2}$.

9. Compute $\left[\begin{matrix} 4 \\ 3 \end{matrix} \right]$ by listing all the appropriate permutations in cycle form.

10. Compute $\left[\begin{matrix} 4 \\ 2 \end{matrix} \right]$ using the recursion formula.

11. Prove Theorem 5.4.3: $\left[\begin{matrix} n \\ k \end{matrix} \right] = (n-1) * \left[\begin{matrix} n-1 \\ k \end{matrix} \right] + \left[\begin{matrix} n-1 \\ k-1 \end{matrix} \right]$.

12. Prove $\left[\begin{matrix} n \\ n-2 \end{matrix} \right] = 2\binom{n}{3} + \frac{\binom{n}{2}\binom{n-2}{2}}{2}$.

13. Construct the following table with entries for $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$.

k / n	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

14. Construct the following table with entries for $\left[\begin{matrix} n \\ k \end{matrix} \right]$.

k / n	1	2	3	4	5	6	7	8	9	10
1										
2										
3										
4										
5										
6										
7										
8										
9										
10										

Let $|D| = n$ and $|R| = n-1$ where n is a finite integer. How many onto functions $f : D \to R$ exist.

Let $|D| = n$ and $|R| = n-2$ where n is a finite integer. How many onto functions $f : D \to R$ exist.