Final Exam Math 2306 sec. 2

Summer 2009

Name: (2 points) Solutions

Your signature (required) confirms that you agree to practice academic honesty.

Signature: ______________________________

Problem	Points
1	
2	
3	
4	
5	
6	
7	
8	
EC	

INSTRUCTIONS: There are 8 problems with the points listed by each one. There is an additional extra credit problem worth 10 points. You may use one sheet (front and back) of notes you prepared.

To receive full credit, you must clearly justify your answer.

(1) (10 points) Use the Laplace transform to solve the initial value problem.

$$y' + y = f(t) \quad y(0) = 1 \quad f(t) = \begin{cases} 0, & 0 \le t < 1 \\ 2, & t \ge 1 \end{cases} = 2\mathcal{U}(t-1)$$

$$\mathcal{L}\{y' + y\} = \mathcal{L}\{2\mathcal{U}(t-1)\}$$

$$sY(s) - y(0) + Y(s) = \frac{2e^{-s}}{s}$$

$$(s+1)Y(s) = \frac{2e^{-s}}{s} + 1 \Rightarrow Y(s) = \frac{2e^{-s}}{s(s+1)} + \frac{1}{s+1}$$

Partial fraction decomp:

$$\frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow 2 = A(s+1) + Bs$$

Let $s = 0$ $\quad 2 = A$

$s = -1$ $\quad 2 = -B$

$$Y(s) = \frac{2e^{-s}}{s} - \frac{2e^{-s}}{s+1} + \frac{1}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= 2\mathcal{U}(t-1) - 2e^{-(t-1)}\mathcal{U}(t-1) + e^{-t}$$

(2) (12 points) Solve the first order initial value problem. An implicit solution is acceptable.

$$\frac{dy}{dx} = \frac{x}{\cos(2y)} \quad y(1) = 0$$

This is separable

$$\int \cos 2y \, dy = \int x \, dx$$

$$\frac{1}{2} \sin 2y = \frac{x^2}{2} + C \implies$$

$$\sin 2y = x^2 + k \qquad \text{where} \quad k = 2C$$

Applying the initial condition

$$\sin(2 \cdot 0) = 1^2 + k \implies 0 = 1 + k \implies k = -1$$

The solution to the IVP is given by

$$\sin 2y = x^2 - 1$$

(3) (12 points) Find the general solution to the nonhomogeneous differential equation.

$y''+y'-6y=-36x$

Solve $y''+y'-6y=0$ for y_c

$m^2+m-6=0 \Rightarrow (m+3)(m-2)=0$

$m=-3$ or $m=2$

$y_c = c_1 e^{-3x} + c_2 e^{2x}$

Using Undetermined coefficients, guess $y_p = Ax+B$

$y_p' = A$, $y_p'' = 0$

$y_p'' + y_p' - 6y_p = -36x$

$A - 6(Ax+B) = -36 \Rightarrow \left.\begin{array}{l} -6A = -36 \\ A - 6B = 0 \end{array}\right\} \Rightarrow \begin{array}{l} A=6 \\ B=1 \end{array}$

The general solution is

$y = c_1 e^{-3x} + c_2 e^{2x} + 6x + 1$

(4) (12 points) Find the form of the particular solution using the method of undetermined coefficients. **Note:** You are only to find the *form* of the solution.

$y''+y'-6y=\sin(3x)+2x^2e^{2x}$ Call $g_1(x)$ $\sin 3x$, $g_2(x) = 2x^2 e^{2x}$

$y_{p_1} = A\sin 3x + B\cos 3x$ (this works)

1st try $y_{p_2} = (Ax^2+Bx+C)e^{2x}$ Ce^{2x} duplicates y_c

corrected $y_{p_2} = (Ax^2+Bx+C)xe^{2x}$

Finally

$y_p = A\sin 3x + B\cos 3x + (Cx^3+Dx^2+Ex)e^{2x}$

(5) (15 points) Find the half range sine and cosine series for the function

$f(x) = 1-x, \quad 0 < x < 1.$ Here $p = 1$

Sine: $b_n = \frac{2}{1}\int_0^1 f(x)\sin\left(\frac{n\pi x}{1}\right)dx = 2\int_0^1 (1-x)\sin(n\pi x)\,dx$

$= 2\left[-\frac{(1-x)}{n\pi}\cos(n\pi x)\Big|_0^1 - \frac{1}{n\pi}\int_0^1 \cancel{\cos(n\pi x)\,dx}\right]$ $\quad u = 1-x, \; du = -dx$

0 $\quad dv = \sin(n\pi x)dx$

$v = \frac{-1}{n\pi}\cos(n\pi x)$

$= 2\left[0 - \frac{-1}{n\pi}\cos 0\right] = \frac{2}{n\pi}$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x)}$$

Cosine: $a_0 = \frac{2}{1}\int_0^1 f(x)dx = 2\int_0^1 (1-x)dx = 2\left[x - \frac{x^2}{2}\right|_0^1 = 2\left(1-\frac{1}{2}\right) = 1$

$a_n = \frac{2}{1}\int_0^1 f(x)\cos\left(\frac{n\pi x}{1}\right)dx = 2\int_0^1 (1-x)\cos(n\pi x)\,dx$

$\quad u = 1-x \quad du = -dx$

$= 2\left[\cancel{\frac{1-x}{n\pi}\sin(n\pi x)}\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \sin(n\pi x)\,dx\right]$ $\quad dv = \cos(n\pi x)dx$

0 $\quad v = \frac{1}{n\pi}\sin(n\pi x)$

$= \frac{-2}{n^2\pi^2}\cos(n\pi x)\Big|_0^1 = \frac{-2}{n^2\pi^2}\left(\cos(n\pi) - \cos 0\right)$

$= \frac{2}{n^2\pi^2}\left(1-(-1)^n\right)$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2(1-(-1)^n)}{n^2\pi^2}\cos(n\pi x)}$$

(6) (12 points) Find the general solution of the linear equation.

$x^2y'+xy=1$

Standard form

Let's assume $x>0$

$y' + \frac{1}{x} y = \frac{1}{x^2}$ $\quad P(x) = \frac{1}{x}$ $\quad \mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$

$$\frac{d}{dx}[xy] = x \cdot \frac{1}{x^2} = \frac{1}{x}$$

$$\int \frac{d}{dx}[xy]\, dx = \int \frac{1}{x} dx = \ln x + C$$

$$xy = \ln x + C$$

$$y = \frac{\ln x + C}{x}$$

(7) (10 points)Use reduction of order and the one given solution to find the general solution to the homogeneous equation.

$x^2y''+xy'-y=0, \quad y_1(x)=x$ $\quad$ Standard form $\quad y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 0$

$$P(x) = \frac{1}{x}, \quad e^{-\int P(x) dx} = e^{-\int \frac{1}{x} dx} = e^{-\ln x} = x^{-1}$$

$$u = \int \frac{e^{-\int P(x) dx}}{y_1^2} dx = \int \frac{x^{-1}}{x^2} dx = \int x^{-3} dx = \frac{x^{-2}}{-2}$$

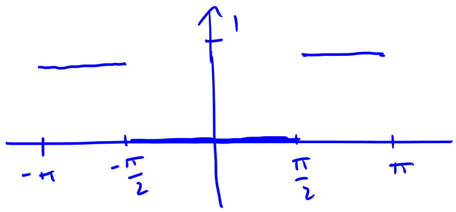
we can multiply by -2 to get rid of the factor $-\frac{1}{2}$.

Take $y_2 = x^{-2} \cdot y_1 = x^{-2} \cdot x = x^{-1}$

i.e. $y_2 = \frac{1}{x}$

(8) (15 points) Give a rough plot of the given function. Determine if it is even, odd or neither, and find its Fourier series.

$$f(x) = \begin{cases} 1, & -\pi < x < \frac{-\pi}{2} \\ 0, & \frac{-\pi}{2} \le x \le \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi \end{cases}$$



Plot of f

It's clearly even.

Its series will not contain any sine terms - i.e. $b_n = 0$ for all n

by symmetry $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi} dx = \frac{2}{\pi}\, x \Big|_{\pi/2}^{\pi} = \frac{2}{\pi}\left[\pi - \frac{\pi}{2}\right] = \frac{2}{\pi}\left(\frac{\pi}{2}\right) = 1$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{\pi/2}^{\pi}\cos(nx)\,dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\sin(nx)\Big|_{\pi/2}^{\pi}\right] = \frac{2}{\pi}\left[\underbrace{\frac{1}{n}\sin(n\pi)}_{0} - \frac{1}{n}\sin\left(n\frac{\pi}{2}\right)\right]$$

$$= \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\boxed{f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(nx)}$$

Extra Credit (10 points): Consider the function f given with its Fourier Series.

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$$

$$f(x) = \frac{\pi^2}{6} + \sum_{n=1}^{\infty} \left\{ \frac{2(-1)^n}{n^2} \cos(nx) + \left(\frac{\pi(-1)^{n+1}}{n} + \frac{2}{\pi n^3} \left[(-1)^n - 1 \right] \right) \sin(nx) \right\}$$

Show that

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

Hint: Consider $f(0)$.