Introduction to Fourier Series

By

Abstract. This paper gives a brief introduction to Fourier series. It emphasizes both conceptual and practical level understanding of some basic Fourier related results, which involves the definition of orthogonal functions, derivation of Fourier coefficients, convergence of Fourier Series and a few practical examples which illustrate the basic steps required to find Fourier Series of known functions.

Introduction

Fourier series is an expansion of a periodic function *f(x)* in terms of an infinite sum of sines and cosines [1]. It ~~is~~ was first introduced by French mathematician Joseph Fourier while he was trying to solve heat conduction problems [2]. Later on, people adopted this technique to approximate periodic functions, analyze radio signals, voltage waveforms, acoustics or some other real world problems in science and engineering fields. Those applications are somewhat complicated and wouldn't be approached properly without knowing basics about Fourier series. So I would love to introduce a few Fourier series properties and related concepts in the next few sections. tense agreement

Orthogonal functions

Definition: Two non-zero functions *f(x)* and *g(x)* are orthogonal over the interval $a \le x \le b$ if $\int_a^b f(x)g(x)dx = 0$ [3].

For this particular case, in order to derive Fourier Series coefficents, we need to make use of orthogonality relationships of sine and cosine functions, which are

$$\int_{-\pi}^{\pi} sin(mx)\, cos(nx)dx = 0 \text{ [4]}$$

when m ≠ n,

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\,dx = 0 \text{ [4]}$$

$$\int_{-\pi}^{\pi} sin(mx) sin(nx)\,dx = 0 \text{ [4]}$$

Here are some trignometric identities,

$sin(mx)\,cos(nx) = \frac{1}{2}\,[sin((m+n)x) + sin((m\text{-}n)x)]$ [4]

$cos(mx)\,cos(nx) = \frac{1}{2}\,[cos((m+n)x) + cos((m\text{-}n)x)]$ [4]

$sin(mx) sin(nx) = \frac{1}{2}\,[cos((m\text{-}n)x) - cos((m+n)x)]$[4]

Knowing these properties, it'd be easy to show the results listed above if we do those integrations. The results of these three integrations are important as we would go on to the next section to derive Fourier coefficients.

Derivation of Fourier coefficients

Suppose we have a function f that is well defined on the interval [$-\pi$, π] and it has 2π, period ~~period~~. Now we want to approximate this function f by a sum of sine and cosine functions. As is ~~studies~~ studied before, $f(x)$ could be written as what it ~~shows~~ show below:

$$f(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2\cos 2x + b_2 \sin 2x) + \cdots$$

$$+ (a_n \cos nx + b_n \sin nx) + \cdots\cdots \text{ [5]}$$

It goes on and on forever.

Now we let n be a finite number, we have

$$f_n(x) = \frac{a_0}{2} + (a_1\cos x + b_1 \sin x) + (a_2\cos 2x + b_2 \sin 2x) + \cdots$$

$$+ (a_n \cos nx + b_n \sin nx)\,,$$

which can be rewritten with sigma notation,

$$f_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}(a_k\,cos\,kx + b_k\,sin\,kx) \qquad \textbf{(I)}$$

Integrate function $f_n(x)$ from $-\pi$ *to* π, we have:

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}[\frac{a_0}{2} + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)]dx$$

$$= \int_{-\pi}^{\pi}\frac{a_0}{2}dx + \int_{-\pi}^{\pi}\sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)\,dx$$

$$= \int_{-\pi}^{\pi}\frac{a_0}{2}dx + \sum_{k=1}^{n}\int_{-\pi}^{\pi}(a_k \cos kx + b_k \sin kx)dx$$

Since *coskx* and *sinkx* are periodic functions on interval $(-\pi, \pi)$, so integrals of *coskx* and *sinkx* over $(-\pi, \pi)$ are both zero. Namely, $\sum_{k=1}^{n}\int_{-\pi}^{\pi}(a_k \cos kx + b_k \sin kx)dx$ is zero.

So,

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi}\frac{a_0}{2}dx = \frac{a_0}{2}[\pi - (-\pi)] = a_o\pi$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

This is the first constant in the Fourier series. Now, we need to do another two similar integrations to obtain general coefficient a_k and b_k. Subscript k is any positive integer.

Multiply both sides of equation **(I)** by *coskx*, we obtain

$$f_n(x)coskx = \frac{a}{2}coskx + [\sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)]coskx$$

$$= \frac{a}{2}coskx + a_1cosx\cdot coskx + b_1sinx\cdot coskx + a_2cos2xcoskx + b_2sin2x\cdot coskx + \cdots$$
$$+ a_kcoskx\cdot coskx + b_ksinkx\cdot coskx + \cdots + a_ncosnx\cdot coskx + a_ksin(nx)\cdot coskx\ .$$

Integrate $f_n(x)coskx$ from $-\pi$ to π, we could see all terms but $a_kcoskx\cdot coskx$ die out (recall orthogonal functions). So, we have

$$\int_{-\pi}^{\pi} f_n(x)coskxdx = \int_{-\pi}^{\pi} a_k coskx \cdot coskx\,dx = \int_{-\pi}^{\pi} a_k(coskx)^2dx$$

Subsequently, we apply trigonometric double-angle formula, which is $cos^2kx = \frac{1+cos2kx}{2}$.

Then, the equation above becomes

$$\int_{-\pi}^{\pi} a_k\left(\frac{1}{2} + \frac{cos2kx}{2}\right)dx = \frac{a_k}{2}[\pi - (-\pi)] = \pi a_k$$

$$\text{So, } a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)coskxdx$$

Likewise, multiply both sides of equation (I) by *sinkx*, we obtain

$f_n\ (x)sinkx = \frac{a}{2}sinkx + [\sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)]sinkx$

$= \frac{a}{2}sinkx + a_1cosx \cdot sinkx + b_1sinx \cdot sinkx + a_2cos2xsinkx + b_2sin2x \cdot sinkx + \cdots +$ $a_kcoskx \cdot sinkx + b_ksinkx \cdot sinkx + \cdots + a_ncosnx \cdot sinkx + a_ksin(nx) \cdot sinkx$.

Integrate $f_n\ (x)sinkx$ from $-\pi$ to π, as indicated above, all terms but $b_ksinkx \cdot sinkx$ vanish this time.

$$\int_{-\pi}^{\pi} f_n(x)sinkxdx = \int_{-\pi}^{\pi} b_ksinkx \cdot sinkx\ dx = \int_{-\pi}^{\pi} b_k(sinkx)^2dx$$

Apply the other trigonometric double-angle formula, $sin^2kx = \frac{1-cos2kx}{2}$.

Then, we have $\int_{-\pi}^{\pi} b_k(coskx)^2dx = \int_{-\pi}^{\pi} b_k\left(\frac{1}{2} - \frac{cos2kx}{2}\right)dx = \frac{b_k}{2}[\pi - (-\pi)] = \pi b_k$

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} fn\ (x)sinkxdx$$

always?

As the definition indicated above, if n approaches infinity, $f(x)$ would be exactly equal to $\frac{a}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$. So we replace $f_n(x)$ with $f(x)$, the general coefficients of Fourier Series are

$$\boldsymbol{a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)coskxdx} \quad \textbf{(i)}$$

$$\boldsymbol{b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinkxdx} \quad \textbf{(ii)}$$

As this block material is studied by many others, for a function whose period isn't $[-\pi, \pi]$ but $[-L, L]$ or $[0, 2L]$, we have similar results.

Theorem: If f is a piecewise continuous function on $[-L, L]$, its Fourier series is

$a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$, where $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$ and for $n \geq 1$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\frac{n\pi x}{L})dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\frac{n\pi x}{L})dx$ [6].

Now, we move further and look at one example that which generally illustrates how to find Fourier series for a specific periodic function.

Ex. $f(x) = e^x \; (0 \le x \le 2\pi)$

First, we need to find the first constant a_0, which formula (i) listed above would apply here. In the meantime, we need to adjust lower and upper limit of integration to 0 and 2π, respectively.

Then we have

$$a_o = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} e^x\,dx = \frac{1}{\pi}\left(e^{2\pi} - 1\right) \qquad \cdots\; k = 0$$

Next step is to find general formula for a_k and b_k.

Apply formula (i) one more time,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x) coskx dx = \frac{1}{\pi}\int_0^{2\pi} e^x coskx dx$$

It's obvious to apply the technique of integration by parts at this point, we can do it separately.

$\int e^x coskxdx = \dfrac{e^x\, coskx + ke^x sinkx}{1+k^2}$, then

$$a_k = \frac{1}{\pi}\int_0^{2\pi} e^x coskxdx = \frac{1}{\pi}\left[\frac{e^x\, coskx + ke^x sinkx}{1+k^2}\right]_0^{2\pi} = \frac{e^{2\pi} - 1}{\pi(1+k^2)}$$

Similarly, we grab formula (ii) to find b_k this time,

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x) sinkxdx = \frac{1}{\pi}\int_0^{2\pi} e^x sinkxdx$$

By applying the same method we used while trying to find coefficient a_k, integration by parts, we could obtain $b_k = \dfrac{k(1 - e^{2\pi})}{\pi(1+k^2)}$.

So $f(x) = e^x \; (0 \le x \le 2\pi)$ could be expanded as a Fourier Series,

$$f(x) = \frac{a}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$$

$$= \frac{1}{2\pi}\left(e^{2\pi} - 1\right) + \sum_{k=1}^{\infty}\left(\frac{e^{2\pi} - 1}{\pi(1+k^2)}\cos kx + \frac{k(1 - e^{2\pi})}{\pi(1+k^2)}\sin kx\right).$$

diction

Convergence of Fourier series

The function in the example we solved above is nice because it's continuous everywhere on its interval, so we don't need to deal with discontinuity problems. But from our experience, we know there are tons of functions that have discontinuous points within ~~its~~ their intervals. Considering this, I'd love to use another example to introduce a new concept, convergence of Fourier series.

$$\text{Ex. } f(x) = \begin{cases} -\frac{\pi}{2}, & -\pi \le x < 0 \\ \frac{\pi}{2}, & 0 \le x < \pi \end{cases}$$

If we follow step by step to obtain a_o, a_k, b_k, we would be able to expand f(x)as follow :

$$f(x) = \sum_{k=1}^{\infty} \frac{1-(-1)^k}{k} sinkx = 2\, sinx + \frac{2}{3} sin3x + \frac{2}{5} sin5x + \cdots\cdots$$

In this case, it turns out the first constant a_0 and general coefficient a_k are both zero. As we notice at point $x= 0$, where function $f(x)$ is discontinuous, all the terms in Fourier series of function $f(x)$ are gone since $sin0$ is zero. So the value of this Fourier series at point $x=0$ is zero. However, it is not what is indicated above from original function$f(x)$, since $f(0)$ is actually $\frac{\pi}{2}$. But this is the typical behavior of Fourier series at discontinuous points of the original function. Let's look at the concept, convergence of Fourier series.

Fourier Convergence Theorem[7]:

Suppose f and f' are two piecewise continuous functions on interval $-L \le x \le L$. The Fourier series of $f(x)$ converges to the periodic extension of $f(x)$ if periodic extension is continuous. On the other hand, in the cases that the function has a jump discontinuity at x=a, its Fourier Series converges to the average of the two one-sided limits, which is $\frac{1}{2}[f(a^-) + f(a^+)]$.

Discussion

what is it?

These are a few fundamental features ~~about~~ of Fourier series. It actually can be further elaborated and combined with applications related to differential equation. Besides, by looking at typical waves like square waves, sawtooth waves and triangle waves, we could further comprehend the analysis value of Fourier series. In addition, Fourier series shortcomings, Gibbs Phenomena, is also a block of interesting supplementary material. But those topics are relatively advanced and all based upon results this paper covers. Through my research, I feel Fourier series can get fairly deep and abstract but its practical value is certainly undoubtful.

what is this?

Reference

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