

## Supplemental Pages

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Series

AP<sub>E</sub>X **CALCULUS II**  
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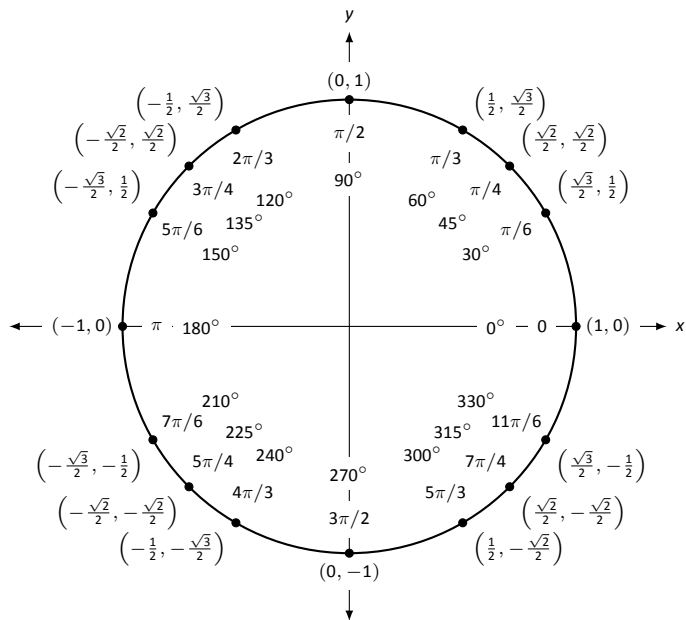
## Differentiation Rules

1.  $\frac{d}{dx}(cx) = c$
2.  $\frac{d}{dx}(u \pm v) = u' \pm v'$
3.  $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4.  $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5.  $\frac{d}{dx}(u(v)) = u'(v)v'$
6.  $\frac{d}{dx}(c) = 0$
7.  $\frac{d}{dx}(x) = 1$
8.  $\frac{d}{dx}(x^n) = nx^{n-1}$
9.  $\frac{d}{dx}(e^x) = e^x$
10.  $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11.  $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12.  $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13.  $\frac{d}{dx}(\sin x) = \cos x$
14.  $\frac{d}{dx}(\cos x) = -\sin x$
15.  $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16.  $\frac{d}{dx}(\sec x) = \sec x \tan x$
17.  $\frac{d}{dx}(\tan x) = \sec^2 x$
18.  $\frac{d}{dx}(\cot x) = -\csc^2 x$
19.  $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$
20.  $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$
21.  $\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$
22.  $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$
23.  $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$
24.  $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$
25.  $\frac{d}{dx}(\cosh x) = \sinh x$
26.  $\frac{d}{dx}(\sinh x) = \cosh x$
27.  $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28.  $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29.  $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30.  $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31.  $\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$
32.  $\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}$
33.  $\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}$
34.  $\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$
35.  $\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$
36.  $\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$

## Integration Rules

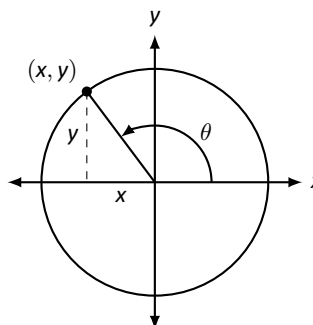
1.  $\int c \cdot f(x) dx = c \int f(x) dx$
2.  $\int f(x) \pm g(x) dx = \int f(x) dx \pm \int g(x) dx$
3.  $\int 0 dx = C$
4.  $\int 1 dx = x + C$
5.  $\int x^n dx = \frac{1}{n+1}x^{n+1} + C, n \neq -1$
6.  $\int e^x dx = e^x + C$
7.  $\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$
8.  $\int \frac{1}{x} dx = \ln|x| + C$
9.  $\int \cos x dx = \sin x + C$
10.  $\int \sin x dx = -\cos x + C$
11.  $\int \tan x dx = -\ln|\cos x| + C$
12.  $\int \sec x dx = \ln|\sec x + \tan x| + C$
13.  $\int \csc x dx = -\ln|\csc x + \cot x| + C$
14.  $\int \cot x dx = \ln|\sin x| + C$
15.  $\int \sec^2 x dx = \tan x + C$
16.  $\int \csc^2 x dx = -\cot x + C$
17.  $\int \sec x \tan x dx = \sec x + C$
18.  $\int \csc x \cot x dx = -\csc x + C$
19.  $\int \cos^2 x dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20.  $\int \sin^2 x dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21.  $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
22.  $\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23.  $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} \sec^{-1}\left(\frac{|x|}{a}\right) + C$
24.  $\int \cosh x dx = \sinh x + C$
25.  $\int \sinh x dx = \cosh x + C$
26.  $\int \tanh x dx = \ln(\cosh x) + C$
27.  $\int \coth x dx = \ln|\sinh x| + C$
28.  $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x + \sqrt{x^2-a^2}| + C$
29.  $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln|x + \sqrt{x^2+a^2}| + C$
30.  $\int \frac{1}{a^2-x^2} dx = \frac{1}{2} \ln\left|\frac{a+x}{a-x}\right| + C$
31.  $\int \frac{1}{x\sqrt{a^2-x^2}} dx = \frac{1}{a} \ln\left(\frac{x}{a + \sqrt{a^2-x^2}}\right) + C$
32.  $\int \frac{1}{x\sqrt{x^2+a^2}} dx = \frac{1}{a} \ln\left|\frac{x}{a + \sqrt{x^2+a^2}}\right| + C$

## The Unit Circle



## Definitions of the Trigonometric Functions

### Unit Circle Definition

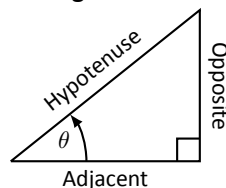


$$\sin \theta = y \quad \cos \theta = x$$

$$\csc \theta = \frac{1}{y} \quad \sec \theta = \frac{1}{x}$$

$$\tan \theta = \frac{y}{x} \quad \cot \theta = \frac{x}{y}$$

### Right Triangle Definition



$$\sin \theta = \frac{O}{H} \quad \csc \theta = \frac{H}{O}$$

$$\cos \theta = \frac{A}{H} \quad \sec \theta = \frac{H}{A}$$

$$\tan \theta = \frac{O}{A} \quad \cot \theta = \frac{A}{O}$$

## Common Trigonometric Identities

### Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

### Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \quad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \quad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \quad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

### Double Angle Formulas

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$= 2 \cos^2 x - 1$$

$$= 1 - 2 \sin^2 x$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

### Sum to Product Formulas

$$\sin x + \sin y = 2 \sin\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2 \sin\left(\frac{x-y}{2}\right) \cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2 \cos\left(\frac{x+y}{2}\right) \cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right)$$

### Power-Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

### Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$

### Product to Sum Formulas

$$\sin x \sin y = \frac{1}{2} (\cos(x-y) - \cos(x+y))$$

$$\cos x \cos y = \frac{1}{2} (\cos(x-y) + \cos(x+y))$$

$$\sin x \cos y = \frac{1}{2} (\sin(x+y) + \sin(x-y))$$

### Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

## Areas and Volumes

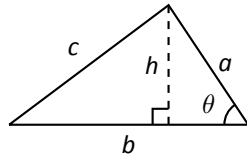
### Triangles

$$h = a \sin \theta$$

$$\text{Area} = \frac{1}{2}bh$$

Law of Cosines:

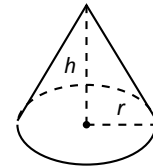
$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



### Right Circular Cone

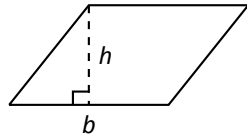
$$\text{Volume} = \frac{1}{3}\pi r^2 h$$

$$\text{Surface Area} = \pi r \sqrt{r^2 + h^2} + \pi r^2$$



### Parallelograms

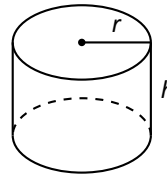
$$\text{Area} = bh$$



### Right Circular Cylinder

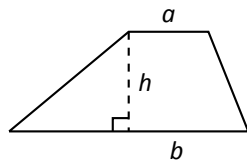
$$\text{Volume} = \pi r^2 h$$

$$\text{Surface Area} = 2\pi rh + 2\pi r^2$$



### Trapezoids

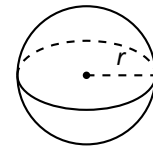
$$\text{Area} = \frac{1}{2}(a + b)h$$



### Sphere

$$\text{Volume} = \frac{4}{3}\pi r^3$$

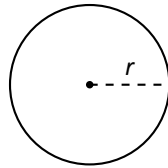
$$\text{Surface Area} = 4\pi r^2$$



### Circles

$$\text{Area} = \pi r^2$$

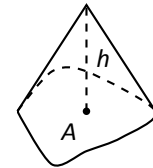
$$\text{Circumference} = 2\pi r$$



### General Cone

$$\text{Area of Base} = A$$

$$\text{Volume} = \frac{1}{3}Ah$$

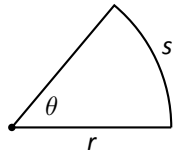


### Sectors of Circles

$\theta$  in radians

$$\text{Area} = \frac{1}{2}\theta r^2$$

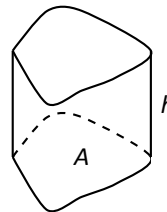
$$s = r\theta$$



### General Right Cylinder

$$\text{Area of Base} = A$$

$$\text{Volume} = Ah$$



# Algebra

## Factors and Zeros of Polynomials

Let  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  be a polynomial. If  $p(a) = 0$ , then  $a$  is a *zero* of the polynomial and a solution of the equation  $p(x) = 0$ . Furthermore,  $(x - a)$  is a *factor* of the polynomial.

## Fundamental Theorem of Algebra

An  $n$ th degree polynomial has  $n$  (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

## Quadratic Formula

If  $p(x) = ax^2 + bx + c$ , and  $0 \leq b^2 - 4ac$ , then the real zeros of  $p$  are  $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

## Special Factors

$$x^2 - a^2 = (x - a)(x + a) \qquad x^3 - a^3 = (x - a)(x^2 + ax + a^2)$$

$$x^3 + a^3 = (x + a)(x^2 - ax + a^2) \qquad x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$$

$$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \dots + nxy^{n-1} + y^n$$

$$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \dots \pm nxy^{n-1} \mp y^n$$

## Binomial Theorem

$$(x + y)^2 = x^2 + 2xy + y^2 \qquad (x - y)^2 = x^2 - 2xy + y^2$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 \qquad (x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 \qquad (x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

## Rational Zero Theorem

If  $p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  has integer coefficients, then every *rational zero* of  $p$  is of the form  $x = r/s$ , where  $r$  is a factor of  $a_0$  and  $s$  is a factor of  $a_n$ .

## Factoring by Grouping

$$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$$

## Arithmetic Operations

$$ab + ac = a(b + c) \qquad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right) \left(\frac{d}{c}\right) = \frac{ad}{bc} \qquad \frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc} \qquad \frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a \left(\frac{b}{c}\right) = \frac{ab}{c} \qquad \frac{a - b}{c - d} = \frac{b - a}{d - c} \qquad \frac{ab + ac}{a} = b + c$$

## Exponents and Radicals

$$a^0 = 1, \quad a \neq 0 \qquad (ab)^x = a^x b^x \qquad a^x a^y = a^{x+y} \qquad \sqrt{a} = a^{1/2} \qquad \frac{a^x}{a^y} = a^{x-y} \qquad \sqrt[n]{a} = a^{1/n}$$

$$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x} \qquad \sqrt[n]{a^m} = a^{m/n} \qquad a^{-x} = \frac{1}{a^x} \qquad \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b} \qquad (a^x)^y = a^{xy} \qquad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

## Additional Formulas

### Summation Formulas:

$$\sum_{i=1}^n c = cn$$
$$\sum_{i=1}^n i = \frac{n(n+1)}{2}$$
$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$
$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

### Trapezoidal Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{2} [f(x_1) + 2f(x_2) + 2f(x_3) + \dots + 2f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^3}{12n^2} [\max |f''(x)|]$$

### Simpson's Rule:

$$\int_a^b f(x) dx \approx \frac{\Delta x}{3} [f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \dots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})]$$

$$\text{with Error} \leq \frac{(b-a)^5}{180n^4} [\max |f^{(4)}(x)|]$$

### Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2} dx$$

### Surface of Revolution:

$$S = 2\pi \int_a^b f(x) \sqrt{1 + f'(x)^2} dx$$

(where  $f(x) \geq 0$ )

$$S = 2\pi \int_a^b x \sqrt{1 + f'(x)^2} dx$$

(where  $a, b \geq 0$ )

### Work Done by a Variable Force:

$$W = \int_a^b F(x) dx$$

### Force Exerted by a Fluid:

$$F = \int_a^b w d(y) \ell(y) dy$$

### Taylor Series Expansion for $f(x)$ :

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

### Maclaurin Series Expansion for $f(x)$ , where $c = 0$ :

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

## Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
<i>n</i> th-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n \rightarrow \infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$ r  < 1$	$ r  \geq 1$	Sum = $\frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n \rightarrow \infty} b_n = L$		Sum = $\left( \sum_{n=1}^a b_n \right) - L$
<i>p</i> -Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n) \, dn$ is convergent	$\int_1^{\infty} a(n) \, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n \rightarrow \infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n \rightarrow \infty} a_n/b_n > 0$	Also diverges if $\lim_{n \rightarrow \infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} < 1$	$\lim_{n \rightarrow \infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n \rightarrow \infty} (a_n)^{1/n} = \infty$