Additional Homework Exercises

Section 5: First Order Equations Models and Applications

(1) An RC-series circuit has resistance R_0 and capacitance C_0. A constant electromotive force of E_0 volts is applied. If the initial charge on the capacitor $q(0) = 0$, find the charge $q(t)$ on the capacitor at time t. Show that the long time charge on the capacitor is the product E_0C_0.

(2) An LR-series circuit has inductance 0.5 h, resistance 10 ohms, and an applied force of $0.1e^{-t}$ volts. If the initial current $i(0) = 1$A, find the current $i(t)$.

(3) A 400 gallon tank contains water into which 10 lbs of salt is dissolved. Salt water containing 3 lbs of salt per gallon is being pumped in at a rate of 4 gallons per minute, and the well mixed solution is being pumped out at the same rate. Let $A(t)$ be the number of lbs of salt in the tank at time t in minutes. Derive the initial value problem governing $A(t)$. Solve this IVP for A.

(4) Suppose the solution in problem (3) is being pumped out at the rate of 5 gallons per minute. Keeping everything else the same as in problem (3), derive the IVP governing A under this new condition. Solve this IVP for A. What is the largest time value for which your solution is physically feasible?

(5) An LR-series circuit has constant inductance and resistance L_0 and R_0, respectively. A constant electromotive force of E_0 is applied. Find the current $i(t)$ if $i(0) = i_0$. Show that for any initial current, the long time current in the circuit is E_0/R_0.

(6) We know that a quantity P that satisfies $\frac{dP}{dt} = P$ experiences exponential growth. The equation $\frac{dP}{dt} = P^{1+\epsilon}$, for $\epsilon > 0$, is refered to as a *doomsday equation*. Solve the following IVP.

$$\frac{dP}{dt} = P^2, \quad P(0) = P_0 \quad \text{where, } P_0 > 0$$

By taking the limit $\lim_{t \to \frac{1}{P_0}^-} P(t)$, deduce the appropriateness of the *doomsday* label.

Solutions

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(1) $q(t) = E_0C_0(1 - e^{-t/R_0C_0}), \quad q \to E_0C_0 \quad \text{as} \quad t \to \infty.$

(2) $i(t) = \dfrac{1}{95}e^{-t} + \dfrac{94}{95}e^{-20t}$

(3) $\dfrac{dA}{dt} + \dfrac{1}{100}A = 12, \quad A(0) = 10. \qquad A(t) = 1200 - 1190e^{-t/100}$

(4) $\dfrac{dA}{dt} + \dfrac{5}{400-t}A = 12, \quad A(0) = 10. \qquad A(t) = 3(400-t) - \dfrac{1190}{400^5}(400-t)^5, \quad 0 \le t \le 400$

(5) $i(t) = \dfrac{E_0}{R_0} + \left(i_0 - \dfrac{E_0}{R_0}\right)e^{-R_0t/L_0}, \quad \lim_{t\to\infty} i(t) = \dfrac{E_0}{R_0} + \left(i_0 - \dfrac{E_0}{R_0}\right)\cdot 0 = \dfrac{E_0}{R_0}$

(6) $P(t) = \dfrac{P_0}{1 - P_0t}, \quad \lim_{t\to\frac{1}{P_0}^-} P(t) = \infty,$

Notice that the solution P exibits *blow up* (i.e. tends to infinity) in finite time.