

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace transform of $f'(t)$. (Assume f is of exponential order c for some c .)

$$\begin{aligned} \mathcal{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t) dt && \text{Int. by parts} \\ &= f(t) e^{-st} \Big|_0^\infty - \int_0^\infty -se^{-st} f(t) dt && u = e^{-st} \quad du = -se^{-st} dt \\ &= 0 - f(0)e^0 + s \underbrace{\int_0^\infty e^{-st} f(t) dt}_{\mathcal{L}\{f(t)\}} && v = f(t) \quad dv = f'(t) dt \\ &= s \mathcal{L}\{f(t)\} - f(0) \end{aligned}$$

Transforms of Derivatives

If $\mathcal{L}\{f(t)\} = F(s)$, we have $\mathcal{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f .

For example

$$\begin{aligned}\mathcal{L}\{f''(t)\} &= s\mathcal{L}\{f'(t)\} - f'(0) \\ &= s(sF(s) - f(0)) - f'(0) \\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y' , y'' and so forth, if

$$\mathcal{L}\{y(t)\} = Y(s),$$

then

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2 Y(s) - sy(0) - y'(0),$$

⋮

$$\mathcal{L}\left\{\frac{d^n y}{dt^n}\right\} = s^n Y(s) - s^{n-1} y(0) - s^{n-2} y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a , b , and c , take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathcal{L}\{y\} = Y(s)$ and $\mathcal{L}\{g(t)\} = G(s)$

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g(t)\}$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{g\}$$

$$a(s^2Y(s) - sy_0 - y'_0) + b(sY(s) - y_0) + cY(s) = G(s)$$

Solve for $Y(s)$ using basic algebra.

$$as^2Y(s) - asy_0 - ay_1 + bsY(s) - by_0 + cY(s) = G(s)$$

$$(as^2 + bs + c)Y(s) - asy_0 - ay_1 - by_0 = G(s)$$

$$(as^2 + bs + c)Y(s) = asy_0 + ay_1 + by_0 + G(s)$$

Characteristic
poly nomial

$$Y(s) = \frac{asy_0 + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

This is $\mathcal{L}\{y(t)\}$ where $y(t)$ is the
solution to the IVP.

To find y , we compute $\mathcal{L}^{-1}\{Y(s)\}$.

Solving IVPs

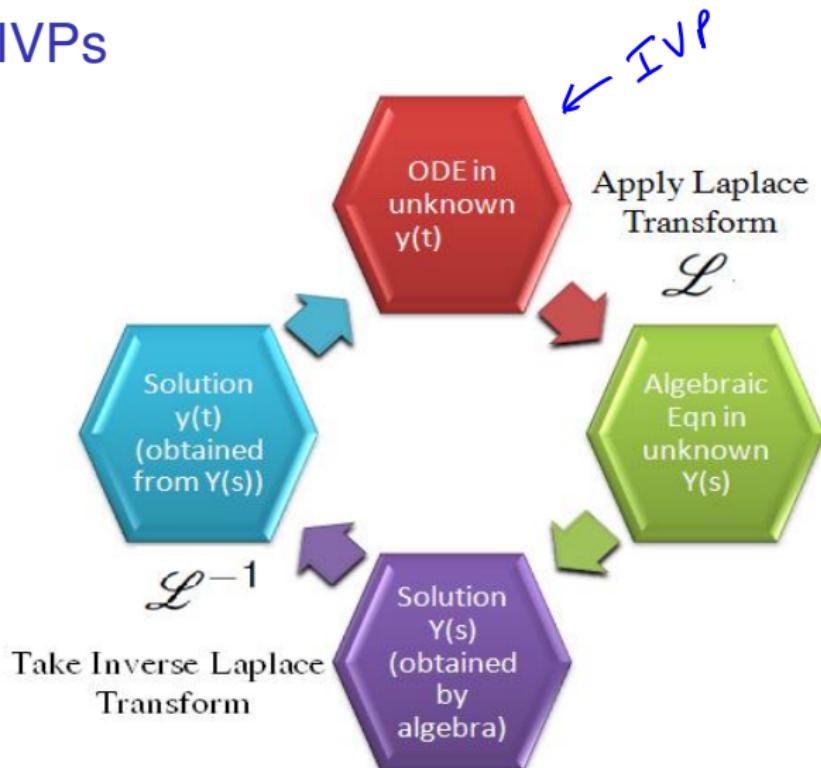


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$\mathcal{L}^{-1} \left\{ \frac{Q(s)}{P(s)} \right\}$ is called the **zero input response**,

and

$\mathcal{L}^{-1} \left\{ \frac{G(s)}{P(s)} \right\}$ is called the **zero state response**.

Solve the IVP using the Laplace Transform

$$(a) \quad \frac{dy}{dt} + 3y = 2t \quad y(0) = 2 \quad \text{Let } \mathcal{L}\{y\} = Y(s)$$

$$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$$

$$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

$$sY(s) - y(0) + 3Y(s) = 2 \left(\frac{1}{s^2}\right) = \frac{2}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2}$$

$$(s+3)Y(s) = 2 + \frac{2}{s^2}$$

$$Y(s) = \frac{2}{s+3} + \frac{\frac{2}{s^2}}{s+3} = \frac{2}{s+3} + \frac{2}{s^2(s+3)}$$

Let's do a partial frac decomp on $\frac{2}{s^2(s+3)}$

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Clear frac
mult by $s^2(s+3)$

$$2 = A s(s+3) + B(s+3) + C s^2$$

$$= A s^2 + 3As + Bs + 3B + Cs^2$$

$$= (A+C)s^2 + (3A+B)s + 3B$$

$$A+C=0 \quad 3B=2 \Rightarrow B=\frac{2}{3} \quad A=-\frac{1}{3}B = -\frac{2}{9}$$

$$3A+B=0$$

$$C = -A = \frac{2}{9}$$

$$Y(s) = \frac{2}{s+3} + \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{2/9}{s+3}$$

$$Y(s) = \frac{\frac{20}{9}}{s+3} - \frac{2/9}{s} + \frac{2/3}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{20}{9} \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \frac{2}{9} \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3} \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

The solution to the IVP is

$$y(t) = \frac{20}{9} e^{-3t} - \frac{2}{9} + \frac{2}{3} t$$

Solve the IVP using the Laplace Transform

$$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0 \quad Y(s) = \mathcal{L}\{y\}$$

$$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$$

$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\} \quad \mathcal{L}\{t\} = \frac{1}{s^2}$$

$$\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$$

$$s^2Y(s) - s\overset{(1)}{y(0)} - \overset{(0)}{y'(0)} + 4\left(sY(s) - \overset{(1)}{y(0)}\right) + 4Y(s) = \frac{1}{(s+2)^2}$$

$$s^2Y(s) - s + 4sY(s) - 4 + 4Y(s) = \frac{1}{(s+2)^2}$$

$$(s^2 + 4s + 4)Y(s) = s + 4 + \frac{1}{(s+2)^2}$$

$$Y(s) = \frac{s+4}{s^2+4s+4} + \frac{1}{s^2+4s+4}$$

$$s^2+4s+4 = (s+2)^2$$

$$= \frac{s+4}{(s+2)^2} + \frac{1}{(s+2)^4}$$

To handle the first term, use

$$\begin{aligned}\frac{s+4}{(s+2)^2} &= \frac{s+2+2}{(s+2)^2} = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} \\ &= \frac{1}{s+2} + \frac{2}{(s+2)^2}\end{aligned}$$

$$Y(s) = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{(s+2)^4}$$

Recall $\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}$

Finally $y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2 \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{3!} \mathcal{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}$

$$y(t) = e^{-2t} + 2t e^{-2t} + \frac{1}{6} t^3 e^{-2t}$$