April 10 Math 2306 sec. 60 Spring 2018

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0, \infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Int by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= f(t)e^{-st}\Big|_0^\infty - \int_0^\infty -se^{-st} f(t)\,dt$$

$$= 0 - f(0)e^0 + s\underbrace{\int_0^\infty e^{-st} f(t)\,dt}_{\mathscr{L}\{f(t)\}}$$

$$= s\mathscr{L}\{f(t)\} - f(0)$$

Transforms of Derivatives

If $\mathscr{L}\left\{f(t)\right\} = F(s)$, we have $\mathscr{L}\left\{f'(t)\right\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\begin{aligned}
\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(sF(s) - f(0)\right) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathcal{L}\{y\} = Y(s)$ and $\mathcal{L}\{g\} = G(s)$

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g\}$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = \mathcal{L}\{g\}$$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

Let's isolate $Y(s)$ using algebra

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)$$

$$(as^2 + bs + c)Y(s) - ay_0 s - ay_1 - by_0 = G(s)$$

$$\underbrace{(as^2 + bs + c)}_{\text{characteristic polynomial}} Y(s) = ay_0 s + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

This is $\mathscr{L}\{y\}$, the solution to the IVP.

To find $y(t)$, take the inverse transform.

$$y(t) = \mathscr{L}^{-1}\left\{Y(s)\right\}.$$

Solving IVPs

IVP

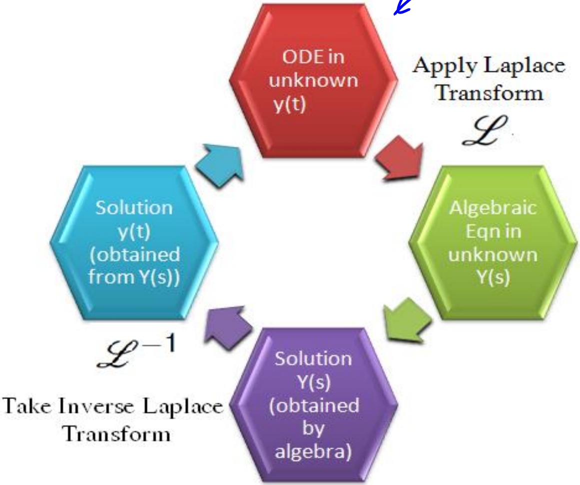

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$

$sY(s) - y(0) + 3Y(s) = 2\left(\dfrac{1!}{s^2}\right) = \dfrac{2}{s^2}$

$(s+3)Y(s) - 2 = \dfrac{2}{s^2}$

$(s+3)Y(s) = 2 + \dfrac{2}{s^2}$

$$Y(s) = \frac{2}{s+3} + \frac{\frac{2}{s^2}}{s+3} = \frac{2}{s+3} + \frac{2}{s^2(s+3)}$$

Partial fraction on $\dfrac{2}{s^2(s+3)}$

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$

Clear frac.
mult. by
$s^2(s+3)$

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$= A(s^2+3s) + B(s+3) + Cs^2$$

$$0s^2 + 0s + 2 = (A+C)s^2 + (3A+B)s + 3B$$

$$\left.\begin{aligned} A+C&=0\\ 3A+B&=0\\ 3B&=2 \end{aligned}\right\} \Rightarrow B=\frac{2}{3},\quad A=\frac{-2}{9},\quad C=\frac{2}{9}$$

$$\frac{2}{s^2(s+3)}=\frac{-2/9}{s}+\frac{2/3}{s^2}+\frac{2/9}{s+3}$$

$$Y(s)=\frac{2}{s+3}-\frac{2/9}{s}+\frac{2/3}{s^2}+\frac{2/9}{s+3}$$

$$=\frac{20/9}{s+3}-\frac{2/9}{s}+\frac{2/3}{s^2}$$

$$y(t)=\frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}-\frac{2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\}+\frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\}$$

$$y(t)=\frac{20}{9}e^{-3t}-\frac{2}{9}+\frac{2}{3}t$$

Solve the IVP using the Laplace Transform

$$y''+4y'+4y = te^{-2t} \quad y(0)=1,\ y'(0)=0$$

$\mathcal{L}\{y\} = Y(s)$

$\mathcal{L}\{y''+4y'+4y\} = \mathcal{L}\{te^{-2t}\}$

$*\ \mathcal{L}\{t\} = \frac{1}{s^2}$

$\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \frac{1}{(s+2)^2}$

$s^2Y(s) - s\underbrace{y(0)}_{=1} - \underbrace{y'(0)}_{=0} + 4\left(sY(s) - \underbrace{y(0)}_{=1}\right) + 4Y(s) = \frac{1}{(s+2)^2}$

$(s^2+4s+4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$

$$(s^2+4s+4)Y(s) = s+4 + \frac{1}{(s+2)^2}$$

$$s^2+4s+4 = (s+2)^2$$

$$Y(s) = \frac{s+4}{(s+2)^2} + \frac{1}{(s+2)^4}$$

Note $\frac{s+4}{(s+2)^2} = \frac{s+2+2}{(s+2)^2} = \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2} = \frac{1}{s+2} + \frac{2}{(s+2)^2}$

$$Y(s) = \frac{1}{s+2} + \frac{2}{(s+2)^2} + \frac{1}{(s+2)^4}$$

$$\mathcal{L}\{t^3\} = \frac{3!}{s^4} \Rightarrow \mathcal{L}\{t^3 e^{-2t}\} = \frac{3!}{(s+2)^4}$$

So $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

$$= \mathscr{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathscr{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\} + \frac{1}{3!}\mathscr{L}^{-1}\left\{\frac{3!}{(s+2)^4}\right\}$$

$$\boxed{y(t) = e^{-2t} + 2te^{-2t} + \frac{1}{6}t^3e^{-2t}}$$