April 11 Math 1190 sec. 62 Spring 2017

Section 5.2: The Definite Integral

Definition (Definite Integral)
Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{c_1, c_2, \ldots, c_n\}$ be any set of sample points. Then the **definite integral of** f **from** a **to** b is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

(3) If f is piecewise continuous enclosing region(s) of total area A_1 **above** the x-axis and enclosing region(s) of total area A_2 **below** the x-axis, then

$$\int_a^b f(x)\,dx = A_1 - A_2$$

For Example...

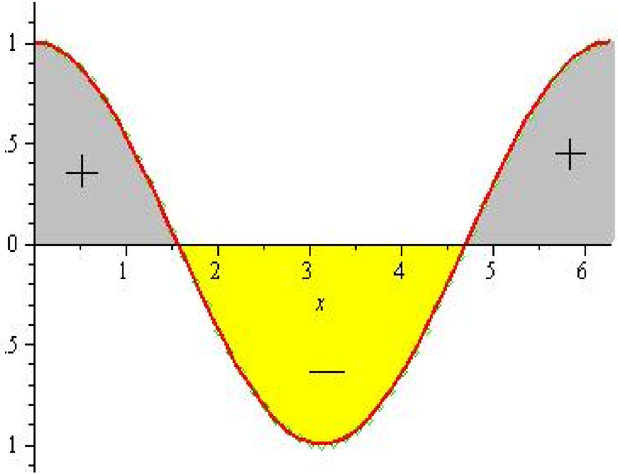
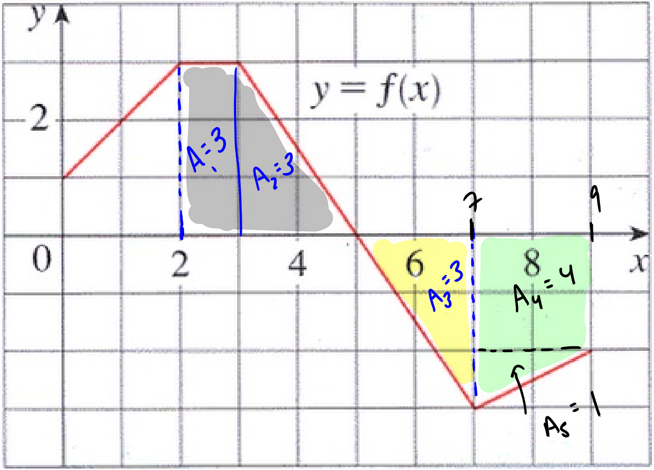


Figure: $\int_a^b f(x)\,dx$ = area of gray region − area of yellow region

Example

Consider the graph of $y = f(x)$ shown.

$A_2 = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$



$A_5 = \frac{1}{2}bh$

$= \frac{1}{2}(1)(2)$

$= 1$

Example

Use the graph on the preceding page to evaluate each integral.

$$\int_2^7 f(x)\,dx = 3+3-3 = 3$$

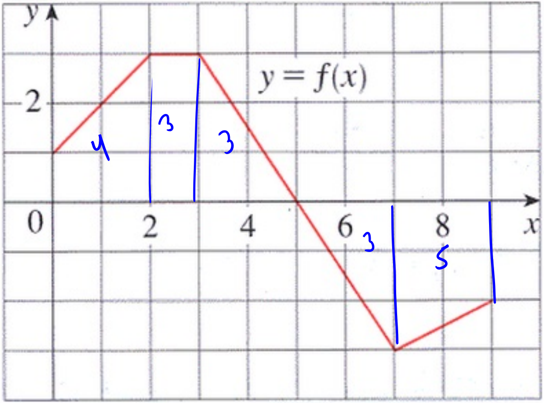
$A_1 + A_2 - A_3$

$$\int_7^9 f(x)\,dx = -5$$

$-(A_4 + A_5)$

Question

Use the graph to evaluate $\int_0^9 f(x)\,dx$



(a) 6

(b) 2

(c) -2

(d) 4

$4+3+3-(3+5)=2$

Important Theorems:

Theorem: If f is continuous on $[a, b]$ or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$

Theorem: If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad c_i = a + i\Delta x.$$

A couple of definitions:

Definition: If $f(a)$ is defined, then

same upper and lower limit

$$\int_a^a f(x)\,dx = 0.$$

In particular, the integral of a continuous function over a single point is zero.

Definition: If $\int_a^b f(x)\,dx$ exists, then

← note the limits are swapped

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Reversing the limits of integration negates the value of the integral.

Example

It can be shown that $\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}$.

Evaluate

$$\int_{\pi}^{0} \sin^2(t)\,dt$$

$$= -\int_0^{\pi} \sin^2(t)\,dt$$

$$= -\frac{\pi}{2}$$

t replacing x doesn't affect the value of the integral.

Swapping limits negates the value

Question

Suppose it is known that $\int_3^{10} f(x)\,dx = -12$

Evaluate $\int_{10}^{3} f(x)\,dx$ $= -\int_3^{10} f(x)\,dx = -(-12) = 12$

(a) 12

(b) -12

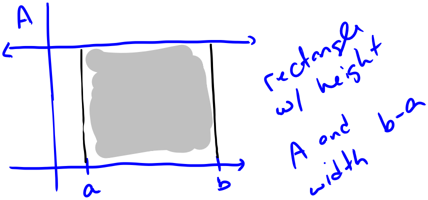
(c) $f(10)$

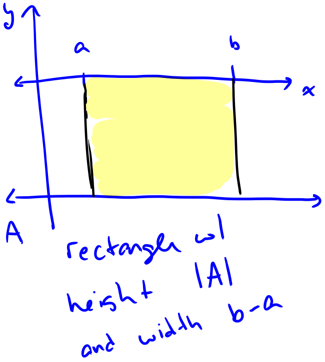
(d) can't be determined without more information

A simple integral

If $f(x) = A$ where A is any constant, then

$$\int_a^b f(x)\,dx = \int_a^b A\,dx = A(b-a).$$





Question

$$\int_3^7 \pi \, dx =$$

$= \pi (7-3) = 4\pi$

(a) 4π

(b) 7π

(c) 3π

(d) can't be determined without more information

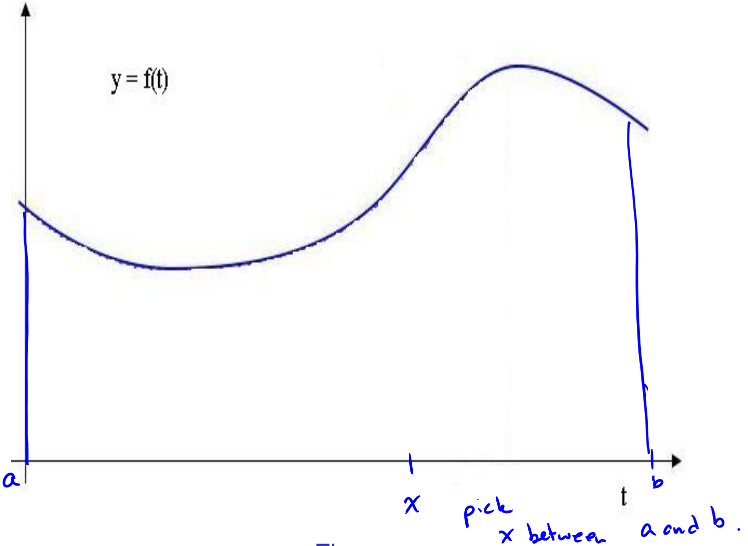
Section 5.3: The Fundamental Theorem of Calculus

Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

$$g(x) = \int_a^x f(t)\, dt$$

How can we understand this function, and what can be said about it?

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



Figure

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a,b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a,b]$ and differentiable on (a,b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a,b)!
"FTC" = "fundamental theorem of calculus"

Example:

Evaluate each derivative.

(a) $\frac{d}{dx}\int_0^x \sin^2(t)\,dt$

(b) $\frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt = \frac{x-\cos x}{x^4+1}$

$f(t) = \frac{t-\cos t}{t^4+1}$

$a = 4$

Question

$$\text{Evaluate} \quad \frac{d}{dx}\int_2^x e^{3t^2}\,dt$$

(a) e^{3x^2}

(b) $6xe^{3x^2}$

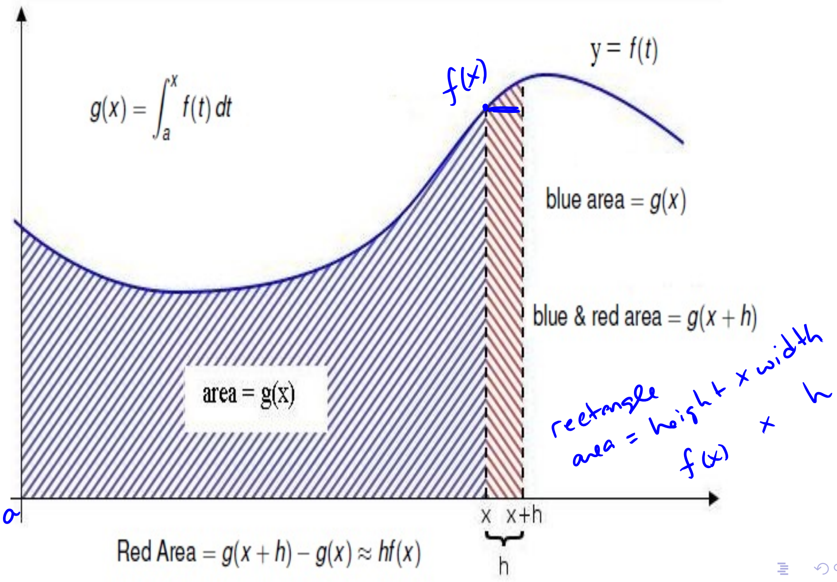
(c) $e^{3x^2} - e^{12}$

$f(t) = e^{3t^2}$

So

$f(x) = e^{3x^2}$

Geometric Argument of FTC



Recall $\quad g'(x) = \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$

$$\text{exact red area} = g(x+h) - g(x)$$

$$\approx h f(x) \leftarrow \text{red rectangle area}$$

$\Rightarrow \quad \dfrac{g(x+h) - g(x)}{h} \approx f(x)$ where the approximation gets better as h gets smaller.

$$\Rightarrow \quad g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} = \lim_{h\to 0} f(x) = f(x)$$

h gets smaller!

Chain Rule with FTC

FTC $\frac{d}{dx}\int_a^x f(x)\,dt = f(x)$

Chain rule

$\frac{d}{dx}F(u) = F'(u)\cdot u'(x)$

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\int_0^{x^2} t^3\,dt$

$\int_0^{x^2} t^3 dt$ is a composition w/ outside $F(u) = \int_0^u t^3 dt \Rightarrow F'(u) = u^3$

and inside $u = x^2 \Rightarrow u'(x) = 2x$

So

$$\frac{d}{dx}\int_0^{x^2} t^3\,dt = \underbrace{(x^2)^3}_{F'(u)} \cdot \underbrace{(2x)}_{u'(x)} = 2x^7$$

(b) $\frac{d}{dx}\int_x^7 \cos(t^2)\,dt$

FTC $\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

The lower limit is constant, the upper is x.

We need the x as the upper limit!

Note

$$\int_x^7 \cos(t^2)\,dt = -\int_7^x \cos(t^2)\,dt$$

So

$$\frac{d}{dx}\int_x^7 \cos(t^2)\,dt = \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt = -\cos(x^2)$$

Question

Use the chain rule where $f(u) = \int_1^u \sin^{-1} t\,dt$ and $u = 7x$ to evaluate

$$\frac{d}{dx}\int_1^{7x} \sin^{-1} t\,dt$$

By the FTC

$f'(u) = \sin^{-1} u$

$u'(x) = 7$

so $f'(u)\cdot u'(x) = \sin^{-1} u \cdot 7$

$= 7\sin^{-1}(7x)$

(a) $\frac{1}{\sqrt{1-7x^2}}$

(b) $\sin^{-1}(7x)$

(c) $\frac{7}{\sqrt{1-49x^2}}$

(d) $7\sin^{-1}(7x)$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

To evaluate $\int_a^b f(x)dx$

- find an antiderivative F of f
- Compute $F(b)$ and $F(a)$
- Then take the difference $F(b) - F(a)$.

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

Here, $a=0$ (lower limit), $b=b$ (the upper limit)

and $f(x) = x$. (i.e. x^1)

Using the power rule, we can take

$$F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$$

$$F(b) = \frac{b^2}{2} \quad \text{and} \quad F(a) = F(0) = \frac{0^2}{2} = 0$$

So $$\int_0^b x\,dx = \frac{b^2}{2} - 0 = \frac{b^2}{2}$$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x\ dx = \frac{x^2}{2}\Big|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

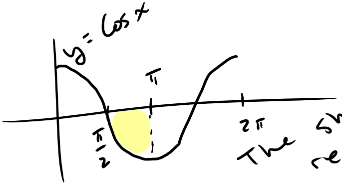
(a) $\int_0^2 3x^2\,dx$ $= x^3\Big|_0^2 = 2^3 - 0^3 = 8 - 0 = 8$

Antiderivative $f(x) = 3x^2$ $F(x) = 3\dfrac{x^3}{3} = x^3$

power rule

(b) $\displaystyle\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = \sin x \Big|_{\frac{\pi}{2}}^{\pi} = \sin\pi - \sin\frac{\pi}{2}$

$= 0 - 1 = -1$



The shaded region has area 1.

Antiderivative $f(x) = \cos x$ $F(x) = \sin x$