April 11 Math 1190 sec. 63 Spring 2017

Section 5.2: The Definite Integral

Definition (Definite Integral)
Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{c_1, c_2, \ldots, c_n\}$ be any set of sample points. Then the **definite integral of f from a to b** is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) If f is positive and continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

(3) If f is piecewise continuous enclosing region(s) of total area A_1 **above** the x-axis and enclosing region(s) of total area A_2 **below** the x-axis, then

$$\int_a^b f(x)\,dx = \; A_1 - A_2$$

For Example...

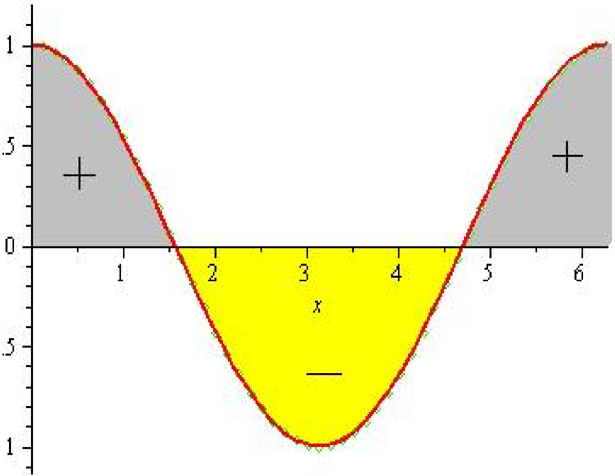
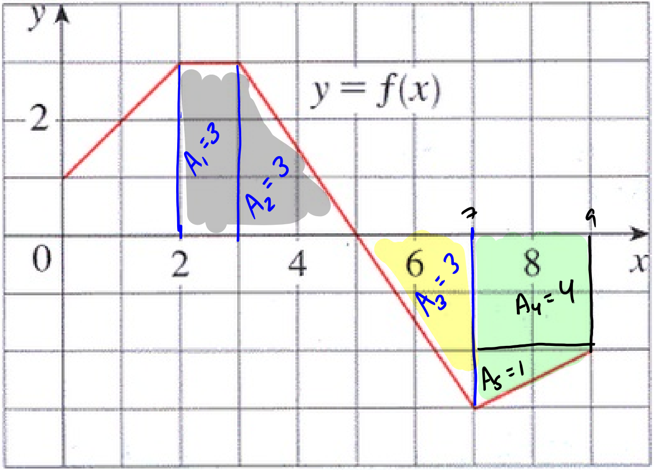


Figure: $\int_a^b f(x)\,dx =$ area of gray region $-$ area of yellow region

Example

Consider the graph of $y = f(x)$ shown.

$A_2 = \frac{1}{2}bh = \frac{1}{2}(2)(3) = 3$



$A_5 = \frac{1}{2}bh$

$= \frac{1}{2}(1)(2)$

$= 1$

Example

Use the graph on the preceding page to evaluate each integral.

$$\int_2^7 f(x)\,dx = 3+3-3=3$$

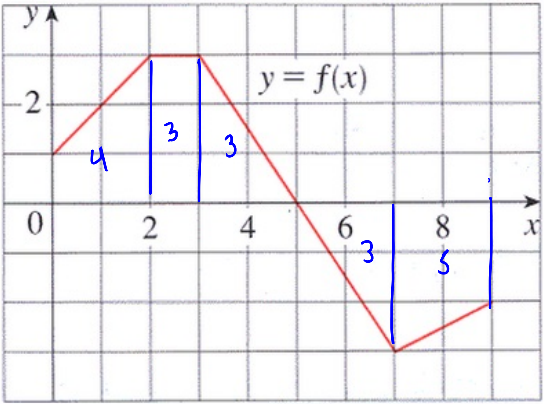
$A_1+A_2-A_3$

$$\int_7^9 f(x)\,dx = -5$$

$-(A_4+A_5)$

Question

Use the graph to evaluate $\displaystyle\int_0^9 f(x)\,dx$



(a) 6

(b) 2

(c) -2

(d) 4

$4+3+3-(3+5)=2$

Important Theorems:

Theorem: If f is continuous on $[a, b]$ or has only finitely many jump discontinuities on $[a, b]$, then f is integrable on $[a, b]$

Theorem: If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x,$$

where

$$\Delta x = \frac{b-a}{n}, \quad \text{and} \quad c_i = a + i\Delta x.$$

A couple of definitions:

same upper and lower limit

Definition: If $f(a)$ is defined, then

$$\int_a^a f(x)\,dx = 0.$$

In particular, the integral of a continuous function over a single point is zero.

Definition: If $\int_a^b f(x)\,dx$ exists, then

limits swapped

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

Reversing the limits of integration negates the value of the integral.

Example

It can be shown that $\int_0^{\pi} \sin^2(x)\,dx = \frac{\pi}{2}$.

Evaluate

$$\int_{\pi}^{0} \sin^2(t)\,dt$$

$$= -\int_0^{\pi} \sin^2(t)\,dt$$

$$= -\frac{\pi}{2}$$

replacing the dummy x with t doesn't affect the value.

switching the limits negates the value.

Question

Suppose it is known that $\int_3^{10} f(x)\,dx = -12$

Evaluate $\int_{10}^{3} f(x)\,dx$ $= - \int_3^{10} f(x)\,dx = -(-12) = 12$

(a) 12

(b) -12

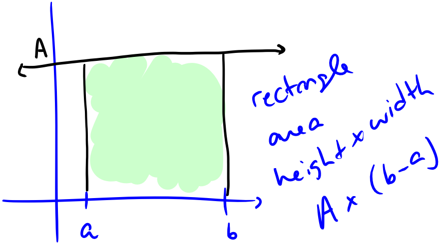
(c) $f(10)$

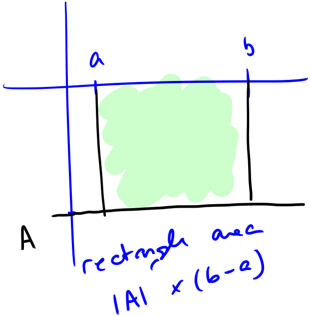
(d) can't be determined without more information

A simple integral

If $f(x) = A$ where A is any constant, then

$$\int_a^b f(x)\,dx = \int_a^b A\,dx = A(b-a).$$





Question

$$\int_3^7 \pi \, dx = \pi(7-3) = 4\pi$$

(a) 4π

(b) 7π

(c) 3π

(d) can't be determined without more information

Section 5.3: The Fundamental Theorem of Calculus

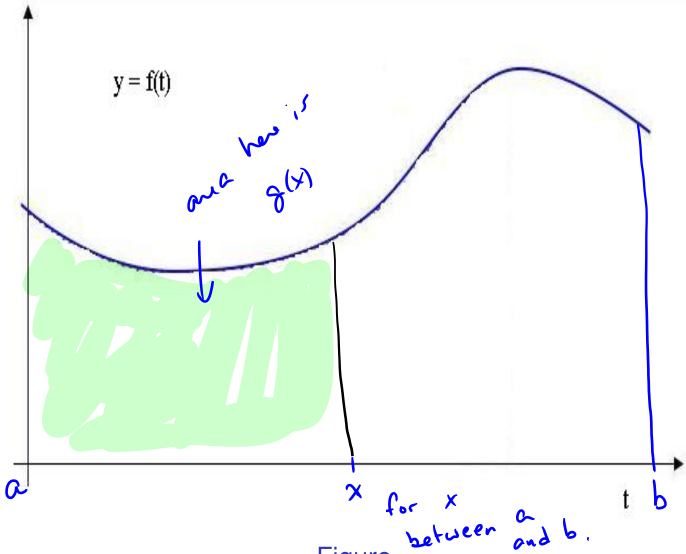
Suppose f is continuous on the interval $[a, b]$. For $a \leq x \leq b$ define a new function

$$g(x) = \int_a^x f(t)\, dt$$

t is a dummy variable, the independent variable is x.

How can we understand this function, and what can be said about it?

Geometric interpretation of $g(x) = \int_a^x f(t)\,dt$



Figure

Theorem: The Fundamental Theorem of Calculus (part 1)

If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

↑ the integrand evaluated @ x.

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Example:

Evaluate each derivative.

Here, $f(t) = \sin^2(t)$

(a) $\displaystyle \frac{d}{dx}\int_0^x \sin^2(t)\,dt = \sin^2(x)$

$a = 0$ and

$g(x) = \displaystyle\int_0^x \sin^2(t)\,dt$

$f(x) = \sin^2(x)$

(b) $\displaystyle \frac{d}{dx}\int_4^x \frac{t-\cos t}{t^4+1}\,dt = \frac{x-\cos x}{x^4+1}$

Here $f(t) = \dfrac{t-\cos t}{t^4+1}$, $a = 4$

Question

$$\text{Evaluate} \quad \frac{d}{dx}\int_2^x e^{3t^2}\,dt$$

(a) e^{3x^2}

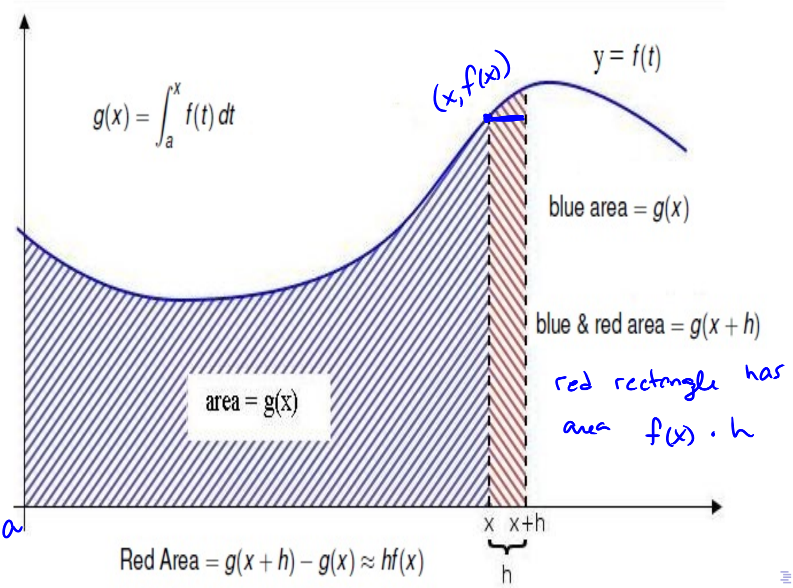
$f(t) = e^{3t^2}$

so $\quad f(x) = e^{3x^2}$

(b) $6xe^{3x^2}$

(c) $e^{3x^2} - e^{12}$

Geometric Argument of FTC



Recall $\quad g'(x) = \lim_{h \to 0} \dfrac{g(x+h) - g(x)}{h}$

$$g(x+h) - g(x) \approx h f(x)$$

$$\Rightarrow \quad \frac{g(x+h) - g(x)}{h} \approx f(x)$$

and the approximation gets better the smaller h is.

$$\Rightarrow \quad \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} = f(x)$$

Chain Rule with FTC

Evaluate each derivative.

(a) $\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3\,dt$

FTC $\dfrac{d}{dx}\displaystyle\int_a^x f(t)\,dt = f(x)$

Chain rule

$\dfrac{d}{dx}F(u(x)) = F'(u)\cdot u'(x)$

This is a composition with inside $u(x) = x^2$

and outside $F(u) = \displaystyle\int_0^u t^3\,dt$

$F'(u) = u^3$ and $u'(x) = 2x$

u^3

$\dfrac{d}{dx}\displaystyle\int_0^{x^2} t^3\,dt = (x^2)^3\cdot(2x)$

$= x^6\cdot 2x \quad = 2x^7$

(b) $\frac{d}{dx}\int_x^7 \cos(t^2)\, dt$

FTC

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

To use the FTC, first note that

$$\int_x^7 \cos(t^2)\,dt = -\int_7^x \cos(t^2)\,dt$$

So

$$\frac{d}{dx}\int_x^7 \cos(t^2)\,dt = \frac{d}{dx}\left(-\int_7^x \cos(t^2)\,dt\right)$$

$$= -\frac{d}{dx}\int_7^x \cos(t^2)\,dt = -\cos(x^2)$$

Question

Use the chain rule where $f(u) = \int_1^u \sin^{-1} t \, dt$ and $u = 7x$ to evaluate

$$\frac{d}{dx}\int_1^{7x} \sin^{-1} t \, dt$$

$f'(u) = \sin^{-1} u$ by the FTC

$u'(x) = 7$

so $f'(u) \cdot u'(x)$

$= (\sin^{-1} u)\, 7 = 7 \sin^{-1}(7x)$

(a) $\frac{1}{\sqrt{1-7x^2}}$

(b) $\sin^{-1}(7x)$

(c) $\frac{7}{\sqrt{1-49x^2}}$

(d) $7\sin^{-1}(7x)$

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a, b]$. (i.e. $F'(x) = f(x)$)

to evaluate $\int_a^b f(x)dx$

- Find an antiderivative $F(x)$.
- Find the numbers $F(b)$ and $F(a)$
- Compute the difference $F(b) - F(a)$

Example: Use the FTC to show that $\int_0^b x\,dx = \frac{b^2}{2}$

Here $a=0$, $b=b$, $f(x)=x$ (x to the 1 power).

An antiderivative is $F(x) = \frac{x^{1+1}}{1+1} = \frac{x^2}{2}$

$F(b) = \frac{b^2}{2}$ and $F(a) = F(0) = \frac{0^2}{2} = 0$

so

$$\int_0^b x\,dx = F(b) - F(a) = \frac{b^2}{2} - 0 = \frac{b^2}{2}$$

Notation

Suppose F is an antiderivative of f. We write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

or sometimes

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b = F(b) - F(a)$$

For example

$$\int_0^b x\ \ dx = \frac{x^2}{2}\Big|_0^b = \frac{b^2}{2} - \frac{0^2}{2} = \frac{b^2}{2}$$

Evaluate each definite integral using the FTC

(a) $\int_0^2 3x^2\,dx = x^3\Big|_0^2 = 2^3 - 0^3 = 8 - 0 = 8$

Side note: $f(x) = 3x^2$, an antiderivative is

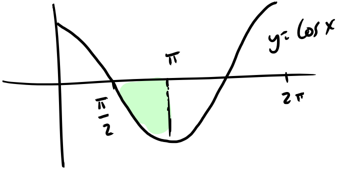
$F(x) = x^3$

$F(x) = 3 \cdot \frac{x^{2+1}}{2+1} = 3\,\frac{x^3}{3} = x^3$

power rule

(b) $\int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = \sin x \Big|_{\pi/2}^{\pi} = \sin \pi - \sin \frac{\pi}{2}$

$= 0 - 1 = -1$



For $f(x) = \cos x$, an antiderivative $F(x) = \sin x$

Question

(c) $\int_1^9 \frac{1}{2} u^{-1/2}\, du = \sqrt{u}\Big|_1^9$

$= \sqrt{9} - \sqrt{1}$

$= 3 - 1 = 2$

$f(u) = \frac{1}{2} u^{-1/2}$, $F(u) = u^{1/2}$

power rule

$\frac{1}{2} \frac{u^{-1/2+1}}{-1/2+1} = \frac{1}{2} \frac{u^{1/2}}{1/2}$

$= u^{1/2}$

(a) 8

(b) $\frac{13}{54}$

(c) 2

(d) $-\frac{1}{3}$

(d) $\displaystyle\int_0^{1/2} \frac{1}{\sqrt{1-t^2}}\,dt = \sin^{-1} t \Big|_0^{1/2} = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0)$

$$= \frac{\pi}{6} - 0 = \frac{\pi}{6}$$

$$f(t) = \frac{1}{\sqrt{1-t^2}}, \quad F(t) = \sin^{-1} t$$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Is this believable? Why or why not?

f is never negative, a negative integral makes no sense! The integral is undefined.

An Observation

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Remember the example: the *area* under the velocity curve gave the net change in position!