## April 12 Math 2306 sec. 57 Spring 2018

## Section 16: Laplace Transforms of Derivatives and IVPs



Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

## Solve the IVP

An LR-series circuit has inductance $L=1$ h, resistance $R=10 \Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0)=0$, find the current $i(t)$ in the circuit.

$$
\begin{aligned}
& \text { urrent } i(t) \text { in the circuit. } \\
& L \frac{d i}{d t}+R_{i}=E(t):
\end{aligned} \quad\left\{\begin{array}{l}
0,0 \leq t<1 \\
E_{0}, 1 \leq t<3 \\
0, t \geqslant 3
\end{array}\right.
$$

LR Circuit Example

$$
\begin{aligned}
& E(t)=0-0 u(t-1)+E_{0} u(t-1)-E_{0} u(t-3)+0 u(t-3) \\
& \frac{d i}{d t}+10 i=E_{0} u(t-1)-E_{0} u(t-3), \quad i(0)=0 \\
& \mathcal{L}\left\{\frac{d i}{d t}+10 i\right\}=\mathcal{L}\left\{E_{0} u(t-1)-E_{0} u(t-3)\right\} \\
& \mathcal{L}\left\{\frac{d i}{d t}\right\}+10 \mathcal{L}\{i\}=E_{0} \mathcal{L}\{u(t-1)\}-E_{0} \mathcal{L}\{u(t-3)\}
\end{aligned}
$$

Let $\mathscr{L}\{i\}=I(s)$

$$
s I(s)-i(0)+10 I(s)=\frac{E_{0}}{s} e^{-s}-\frac{E_{0}}{s} e^{-3 s}
$$

$$
\begin{aligned}
& (s+10) I(s)=\frac{E_{0}}{s} e^{-s}-\frac{E}{s} e^{-3 s} \\
& I(s)=\frac{E_{0}}{s(s+10)} e^{-s}-\frac{E_{0}}{s(s+10)} e^{-3 s}
\end{aligned}
$$

Decompor

$$
\left.\begin{array}{rl}
\frac{E_{0}}{s(s+10)} & =\frac{A}{s}+\frac{B}{s+10} \\
\Rightarrow & E_{0}=A(s+10)+B s \\
s & =0 \quad E_{0}=10 A \quad \Rightarrow \quad A
\end{array}\right)=\frac{E_{0}}{10}, ~ B=\frac{-E_{0}}{10}
$$

$$
I(s)=\left(\frac{E_{0}}{\frac{10}{10}}-\frac{\frac{E_{0}}{10}}{s+10}\right) e^{-s}-\left(\frac{\frac{E_{0}}{1_{0}}}{s}-\frac{\frac{E_{0}}{10}}{s+10}\right) e^{-3 s}
$$

well use

$$
\mathscr{L}^{-1}\left\{e^{-a s} F(s)\right\}=f(t-a) U(t-a)
$$

Also, note

$$
\begin{aligned}
& \mathcal{L}^{-1}\left\{\frac{E_{0}}{10} s-\frac{\frac{E_{0}}{10}}{s+10}\right\}=\frac{E_{0}}{10}-\frac{E_{0}}{10} e^{-10 t} \\
& i(t)=\mathscr{L}^{-1}\{I(s)\}
\end{aligned}
$$

$$
\begin{aligned}
& i(t)=\mathcal{L}^{-1}\left\{\left(\frac{\frac{E_{0}}{10}}{s}-\frac{E_{0}}{10}\right) e^{-s+10}\right\}-\mathscr{L}^{-1}\left\{\left(\frac{\frac{E_{0}}{10}}{s}-\frac{\frac{E_{0}}{10}}{s+10}\right) e^{-3 s}\right\} \\
& i(t)=\left(\frac{E_{0}}{10}-\frac{E_{0}}{10} e^{-10(t-1)}\right) U(t-1)-\left(\frac{E_{0}}{10}-\frac{E_{0}}{10} e^{-10(t-3)}\right) U(t-3)
\end{aligned}
$$

We con write this in pieces

$$
\text { If } 0 \leq t<1 \quad u(t-1)=0 \text { and } u(t-3)=0
$$

$$
\begin{aligned}
& 1 \leq t<3 \quad u(t-1)=1 \quad \text { and } u(t-3)=0 \\
& t \geqslant 3 \quad u(t-1)=1 \text { and } u(t-3)=1
\end{aligned}
$$

$$
i(t)=\left\{\begin{array}{l}
0,0 \leq t<1 \\
\frac{E_{0}}{10}-\frac{E_{0}}{10} e^{-10(t-1)}, \quad 1 \leq t<3 \\
\frac{E_{0}}{10} e^{-10(t-3)}-\frac{E_{0}}{10} e^{-10(t-1)}, \quad t \geqslant 3
\end{array}\right.
$$

## Section 17: Fourier Series: Trigonometric Series

## Some Preliminary Concepts

Suppose two functions $f$ and $g$ are integrable on the interval $[a, b]$. We define the inner product of $f$ and $g$ on $[a, b]$ as

$$
\langle f, g\rangle=\int_{a}^{b} f(x) g(x) d x \in \begin{aligned}
& \text { this } \\
& a
\end{aligned}
$$

We say that $f$ and $g$ are orthogonal on $[a, b]$ if

$$
\langle f, g\rangle=0 .
$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

## Properties of an Inner Product

Let $f, g$, and $h$ be integrable functions on the appropriate interval and let $c$ be any real number. The following hold
(i) $<f, g>=<g, f>$
(ii) $<f, g+h>=<f, g>+<f, h>$
(iii) $<c f, g>=c<f, g>$
(iv) $<f, f>\geq 0$ and $<f, f>=0$ if and only if $f=0$

## Orthogonal Set

A set of functions $\left\{\phi_{0}(x), \phi_{1}(x), \phi_{2}(x), \ldots\right\}$ is said to be orthogonal on an interval $[a, b]$ if

$$
<\phi_{m}, \phi_{n}>=\int_{a}^{b} \phi_{m}(x) \phi_{n}(x) d x=0 \text { whenever } m \neq n .
$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$
\langle\phi, \phi\rangle=\int_{a}^{b} \phi^{2}(x) d x>0 .
$$

Hence we define the square norm of $\phi$ (on $[a, b]$ ) to be

$$
\|\phi\|=\sqrt{\int_{a}^{b} \phi^{2}(x) d x} .
$$

An Orthogonal Set of Functions
Consider the set of functions

$$
\{1, \cos x, \cos 2 x, \cos 3 x, \ldots, \sin x, \sin 2 x, \sin 3 x, \ldots\} \text { on }[-\pi, \pi] .
$$

Evaluate $\langle\cos (n x), 1\rangle$ and $\langle\sin (m x), 1\rangle$.

$$
\begin{aligned}
\langle\cos (n x), 1\rangle & =\int_{-\pi}^{\pi} \cos (n x) \cdot 1 d x=\left.\frac{1}{n} \sin (n x)\right|_{-\pi} ^{\pi} \\
& =\frac{1}{n} \sin (n \pi)-\frac{1}{n} \sin (-n \pi)=0
\end{aligned}
$$

$\sin (n \pi)=0$ for every integer $n$ * $\operatorname{Cos}(n x)$ and 1 are orthogond on $[-\pi, \pi]$.

$$
\begin{aligned}
\langle\sin (m x), 1\rangle & =\int_{-\pi}^{\pi} \sin (m x) \cdot 1 d x \\
& =\left.\frac{-1}{m} \cos (m x)\right|_{-\pi} ^{\pi} \\
& =\frac{-1}{m} \cos (m \pi)-\frac{-1}{m} \cos (-m \pi) \\
& =\frac{-1}{m} \cos (m \pi)+\frac{1}{m} \cos (m \pi)=0
\end{aligned}
$$

* $\sin (m x)$ and 1 are o-thogoned on

$$
[-\pi, \pi] .
$$

## An Orthogonal Set of Functions

Consider the set of functions
$\{1, \cos x, \cos 2 x, \cos 3 x, \ldots, \sin x, \sin 2 x, \sin 3 x, \ldots\}$ on $\quad[-\pi, \pi]$.
It can easily be verified that
$\int_{-\pi}^{\pi} \cos n x d x=0$ and $\int_{-\pi}^{\pi} \sin m x d x=0$ for all $n, m \geq 1$,
$\int_{-\pi}^{\pi} \cos n x \sin m x d x=0$ for all $m, n \geq 1, \quad$ and
$\int_{-\pi}^{\pi} \cos n x \cos m x d x=\int_{-\pi}^{\pi} \sin n x \sin m x d x=\left\{\begin{array}{ll}0, & m \neq n \\ \pi, & n=m\end{array}\right.$,

## An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions
$\{1, \cos x, \cos 2 x, \cos 3 x, \ldots, \sin x, \sin 2 x, \sin 3 x, \ldots\}$
is an orthogonal set on the interval $[-\pi, \pi]$.

## An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t=\frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$
\left\{1, \cos \frac{n \pi x}{p}, \left.\sin \frac{m \pi x}{p} \right\rvert\, n, m \in \mathbb{N}\right\} \underset{\substack{\text { naturel } \\ \text { numbers }}}{\substack{\text { nin }}}
$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

## Fourier Series

Suppose $f(x)$ is defined for $-\pi<x<\pi$. We would like to know how to write $f$ as a series in terms of sines and cosines.

Task: Find coefficients (numbers) $a_{0}, a_{1}, a_{2}, \ldots$ and $b_{1}, b_{2}, \ldots$ such that ${ }^{1}$

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \sin n x\right)
$$

[^0]
## Fourier Series

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \sin n x\right) .
$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$
f(x) \sim \frac{a_{0}}{2}+\cdots
$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient
For a known function $f$ defined on $(-\pi, \pi)$, assume there is such a series ${ }^{2}$. Let's find the coefficient $b_{4}$.
$f$ coff. of $\sin (4 x)$

$$
f(x) \sin (4 x)=\frac{a_{0}}{2} \sin (4 x)+\sum_{n=1}^{\infty}\left(a_{n} \cos n x \sin (4 x)+b_{n} \sin n x \sin (4 x)\right)
$$

tulegoter from $-\pi$ to $\pi$

$$
\begin{aligned}
& \int_{-\pi}^{\pi} f(x) \sin (4 x) d x=\frac{a_{0}}{2} \int_{-\pi}^{\pi} \sin (4 x) d x \\
& \sum_{n=1}^{\infty}\left[a_{n} \int_{-\pi}^{\pi} \cos (n x) \sin (4 x) d x+b_{n} \int_{-\pi}^{\pi} \sin (n x) \sin (4 x) d x\right]
\end{aligned}
$$

${ }^{2}$ We will also assume that the order of integrating and summing can be interchanged.

$$
\begin{aligned}
& \Rightarrow \quad \int_{-\pi}^{\pi} f(x) \sin (4 x) d x=\sum_{n=1}^{\infty} b_{n} \int_{-\pi}^{\pi} \sin (n x) \sin (4 x) d x \\
& \begin{cases}0, & n \neq 4 \\
\pi, & n=4\end{cases} \\
& \Rightarrow \int_{-\pi}^{\pi} f(x) \sin (4 x) d x=\pi b_{4} \\
& \Rightarrow b_{4}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (4 x) d x
\end{aligned}
$$

## Finding Fourier Coefficients

Note that there was nothing special about seeking the $4^{\text {th }}$ sine coefficient $b_{4}$. We could have just as easily sought $b_{m}$ for any positive integer $m$. We would simply start by introducing the factor $\sin (m x)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos (m x)$-including the constant term since $\cos (0 \cdot x)=1$. The only minor difference we want to be aware of is that

$$
\int_{-\pi}^{\pi} \cos ^{2}(m x) d x= \begin{cases}2 \pi, & m=0 \\ \pi, & m \geq 1\end{cases}
$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_{0}}{2}$ as opposed to just $a_{0}$.

## The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The Fourier series of the function $f$ defined on $(-\pi, \pi)$ is given by

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \sin n x\right) .
$$

Where

$$
\begin{aligned}
a_{0} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) d x \\
a_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n x d x, \quad \text { and } \\
b_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x d x
\end{aligned}
$$


[^0]:    ${ }^{1}$ We'll write $\frac{a_{0}}{2}$ as opposed to $a_{0}$ purely for convenience.

