April 12 Math 2306 sec. 60 Spring 2018

Section 16: Laplace Transforms of Derivatives and IVPs

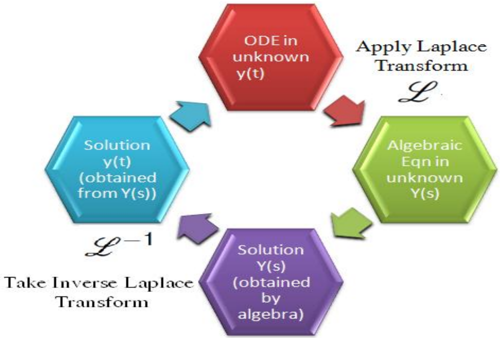


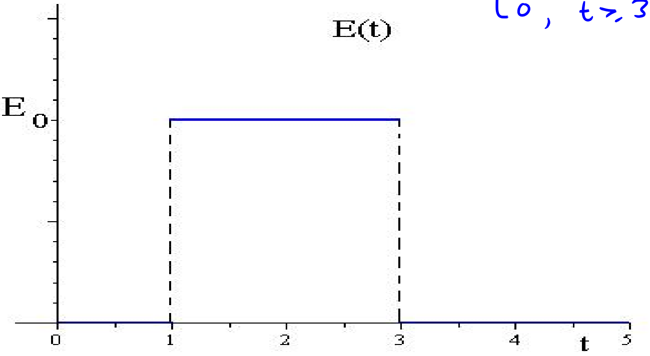
Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

Solve the IVP

An LR-series circuit has inductance $L = 1\text{h}$, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.

$$L\frac{di}{dt} + Ri = E$$

$$E = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \geq 3 \end{cases}$$



LR Circuit Example

$$E(t) = 0 - 0\,\mathcal{U}(t-1) + E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3) + 0\,\mathcal{U}(t-3)$$

$$= E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)$$

$$\frac{di}{dt} + 10i = E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3), \quad i(0) = 0$$

Let $\mathcal{L}\{i(t)\} = I(s)$

$$\mathcal{L}\left\{\frac{di}{dt} + 10i\right\} = \mathcal{L}\left\{E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3)\right\}$$

$$\mathcal{L}\left\{\frac{di}{dt}\right\} + 10\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underset{0}{i(0)} + 10\,I(s) = \frac{E_0}{s} e^{-s} - \frac{E_0}{s} e^{-3s}$$

$$(s+10)\,I(s) = \frac{E_0}{s} e^{-s} - \frac{E_0}{s} e^{-3s}$$

$$I(s) = \frac{E_0}{s(s+10)} e^{-s} - \frac{E_0}{s(s+10)} e^{-3s}$$

Partial frac:

$$\frac{E_0}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$E_0 = A(s+10) + Bs$$

$s = 0 \quad E_0 = 10A \quad A = \frac{E_0}{10}$

$s = -10 \quad E_0 = -10B \quad B = \frac{-E_0}{10}$

$$I(s) = \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-s} - \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-3s}$$

* Recall

$$\mathscr{L}^{-1}\left\{ e^{-as} F(s) \right\} = f(t-a)\,\mathcal{U}(t-a)$$

where $\mathscr{L}^{-1}\{F(s)\} = f(t)$

$$\mathscr{L}^{-1}\left\{ \frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right\} = \frac{E_0}{10}\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{E_0}{10}\mathscr{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= \frac{E_0}{10} - \frac{E_0}{10} e^{-10t}$$

$$i(t) = \mathcal{L}^{-1}\left\{ I(s) \right\}$$

$$= \mathcal{L}^{-1}\left\{ \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-s} \right\} - \mathcal{L}^{-1}\left\{ \left(\frac{\frac{E_0}{10}}{s} - \frac{\frac{E_0}{10}}{s+10} \right) e^{-3s} \right\}$$

$$= \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} \right) \mathcal{U}(t-1) - \left(\frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-3)} \right) \mathcal{U}(t-3)$$

For $0 \le t < 1$ $\quad \mathcal{U}(t-1) = 0$ and $\mathcal{U}(t-3) = 0$

For $1 \le t < 3$ $\quad \mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 0$

For $t \geq 3$ $\quad \mathcal{U}(t-1) = 1$ and $\mathcal{U}(t-3) = 1$

$$
i(t) = \begin{cases} 0 , & 0 \le t < 1 \\ \frac{E_0}{10} - \frac{E_0}{10} e^{-10(t-1)} , & 1 \le t < 3 \\ \frac{E_0}{10} e^{-10(t-3)} - \frac{E_0}{10} e^{-10(t-1)} , & t \geq 3 \end{cases}
$$

Section 17: Fourier Series: Trigonometric Series

Some Preliminary Concepts

Suppose two functions f and g are integrable on the interval $[a, b]$. We define the **inner product** of f and g on $[a, b]$ as

$$< f, g >= \int_a^b f(x)g(x)\, dx. \quad \leftarrow \text{a number}$$

We say that f and g are **orthogonal** on $[a, b]$ if

$$< f, g >= 0.$$

The product depends on the interval, so the orthogonality of two functions depends on the interval.

Properties of an Inner Product

Let f, g, and h be integrable functions on the appropriate interval and let c be any real number. The following hold

(i) $< f, g > = < g, f >$

(ii) $< f, g + h > = < f, g > + < f, h >$

(iii) $< cf, g > = c < f, g >$

(iv) $< f, f > \geq 0$ and $< f, f > = 0$ if and only if $f = 0$

Orthogonal Set

A set of functions $\{\phi_0(x), \phi_1(x), \phi_2(x), \dots\}$ is said to be **orthogonal** on an interval $[a, b]$ if

$$< \phi_m, \phi_n >= \int_a^b \phi_m(x)\phi_n(x)\, dx = 0 \quad \text{whenever} \quad m \neq n.$$

Note that any function $\phi(x)$ that is not identically zero will satisfy

$$< \phi, \phi >= \int_a^b \phi^2(x)\, dx > 0.$$

Hence we define the **square norm** of ϕ (on $[a, b]$) to be

$$\|\phi\| = \sqrt{\int_a^b \phi^2(x)\, dx}.$$

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

Evaluate $\langle \cos(nx), 1\rangle$ and $\langle \sin(mx), 1\rangle$.

$$\langle \cos(nx), 1\rangle = \int_{-\pi}^{\pi} \cos(nx)\cdot 1 \, dx$$

$$= \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} = \frac{1}{n}\sin(n\pi) - \frac{1}{n}\sin(-n\pi)$$

$$= 0 - 0 = 0$$

$\sin(n\pi) = 0$ for all integers n

$\cos(nx)$ and 1 are orthogonal on $[-\pi, \pi]$.

$$\langle \sin(mx), 1 \rangle = \int_{-\pi}^{\pi} \sin(mx)\cdot 1\, dx$$

$$= \frac{-1}{m}\cos(mx)\Big|_{-\pi}^{\pi} = \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(-m\pi)$$

$$= \frac{-1}{m}\cos(m\pi) + \frac{1}{m}\cos(m\pi)$$

$$= 0$$

$$\cos(-\theta) = \cos(\theta)$$

So $\sin(mx)$ and 1 are orthogonal on $[-\pi, \pi]$.

An Orthogonal Set of Functions

Consider the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\} \quad \text{on} \quad [-\pi, \pi].$$

It can easily be verified that

$$\int_{-\pi}^{\pi} \cos nx\, dx = 0 \quad \text{and} \quad \int_{-\pi}^{\pi} \sin mx\, dx = 0 \quad \text{for all} \quad n, m \geq 1,$$

$$\int_{-\pi}^{\pi} \cos nx\, \sin mx\, dx = 0 \quad \text{for all} \quad m, n \geq 1, \quad \text{and}$$

$$\int_{-\pi}^{\pi} \cos nx\, \cos mx\, dx = \int_{-\pi}^{\pi} \sin nx\, \sin mx\, dx = \begin{cases} 0, & m \neq n \\ \pi, & n = m \end{cases},$$

An Orthogonal Set of Functions on $[-\pi, \pi]$

These integral values indicated that the set of functions

$$\{1, \cos x, \cos 2x, \cos 3x, \ldots, \sin x, \sin 2x, \sin 3x, \ldots\}$$

is an orthogonal set on the interval $[-\pi, \pi]$.

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,\middle|\, n, m \in \mathbb{N}\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) a_0, $a_1, a_2, \ldots$ and $b_1, b_2, \ldots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

Fourier Series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

The question of convergence naturally arises when we wish to work with infinite series. To highlight convergence considerations, some authors prefer not to use the equal sign when expressing a Fourier series and instead write

$$f(x) \sim \frac{a_0}{2} + \cdots$$

Herein, we'll use the equal sign with the understanding that equality may not hold at each point.

Convergence will be address later.

Finding an Example Coefficient

For a known function f defined on $(-\pi, \pi)$, assume there is such a series[2]. Let's find the coefficient b_4.
↑ coef. of $\sin(4x)$

$$f(x)\sin(4x) = \frac{a_0}{2}\sin(4x) + \sum_{n=1}^{\infty}\left(a_n \cos nx \sin(4x) + b_n \sin nx \sin(4x)\right).$$

Integrate from $-\pi$ to π

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \frac{a_0}{2}\underbrace{\int_{-\pi}^{\pi}\sin(4x)\,dx}_{=0} +$$

$$\sum_{n=1}^{\infty}\left(a_n \underbrace{\int_{-\pi}^{\pi}\cos(nx)\sin(4x)\,dx}_{=0} + b_n\int_{-\pi}^{\pi}\sin(nx)\sin(4x)\,dx\right)$$

[2]We will also assume that the order of integrating and summing can be interchanged.

$$\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin(nx)\sin(4x)\,dx$$

$$\text{"} \quad 0 \text{ if } n \neq 4$$
$$\pi \text{ if } n = 4$$

$$\Rightarrow \int_{-\pi}^{\pi} f(x)\sin(4x)\,dx = \pi b_4$$

$$\Rightarrow b_4 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(4x)\,dx$$

Finding Fourier Coefficients

Note that there was nothing special about seeking the 4^{th} sine coefficient b_4. We could have just as easily sought b_m for any positive integer m. We would simply start by introducing the factor $\sin(mx)$.

Moreover, using the same orthogonality property, we could pick on the a's by starting with the factor $\cos(mx)$—including the constant term since $\cos(0 \cdot x) = 1$. The only minor difference we want to be aware of is that

$$\int_{-\pi}^{\pi} \cos^2(mx)\, dx = \begin{cases} 2\pi, & m = 0 \\ \pi, & m \geq 1 \end{cases}$$

Careful consideration of this sheds light on why it is conventional to take the constant to be $\frac{a_0}{2}$ as opposed to just a_0.

The Fourier Series of $f(x)$ on $(-\pi,\pi)$

The **Fourier series** of the function f defined on $(-\pi,\pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx, \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and} \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx
\end{aligned}$$