# April 10 Math 3260 sec. 55 Spring 2018

## Section 6.3: Orthogonal Projections

**Theorem:** If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis of a subspace $W$ of $\mathbb{R}^n$, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$ then

$$\text{proj}_W\, \mathbf{y} = \sum_{j=1}^{p} (\mathbf{y} \cdot \mathbf{u}_j)\, \mathbf{u}_j.$$

And, if $U$ is the matrix $U = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_p]$, then the above is equivalent to

$$\text{proj}_W\, \mathbf{y} = UU^T\mathbf{y}.$$

**Remark:** Note that the projection doesn't depend on which basis used as long as it is an orthonormal basis. But this does raise a question about how one might obtain an orthonormal basis.

# Section 6.4: Gram-Schmidt Orthogonalization

**Question:** Given any-old basis for a subspace $W$ of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

**Example:** Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} \right\}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ that spans $W$.

To ensure $\vec{v}_1, \vec{v}_2$ are in W, we write

$$\vec{v}_1 = c_1\vec{x}_1 + c_2\vec{x}_2 \quad \text{and} \quad \vec{v}_2 = d_1\vec{x}_1 + d_2\vec{x}_2$$

We need to choose $c_1, c_2, d_1, d_2$ so that $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Let's let $c_1 = 1$, $c_2 = 0$ so $\vec{v}_1 = \vec{x}_1$.

To ensure we get $\vec{x}_2$ stuff let's set $d_2 = 1$. So

$$\vec{v}_2 = \vec{x}_2 + d_1 \vec{x}_1 = \vec{x}_2 + d_1 \vec{v}_1 \quad \text{since} \quad \vec{v}_1 = \vec{x}_1$$

we insist that $\vec{v}_1 \cdot \vec{v}_2 = 0$

$$\vec{v}_2 \cdot \vec{v}_1 = (\vec{x}_2 + d_1 \vec{v}_1) \cdot \vec{v}_1$$

$$0 = \vec{x}_2 \cdot \vec{v}_1 + d_1 \vec{v}_1 \cdot \vec{v}_1$$

$$\Rightarrow \quad d_1 \vec{v}_1 \cdot \vec{v}_1 = -\vec{x}_2 \cdot \vec{v}_1 \Rightarrow d_1 = \frac{-\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} = \frac{-\vec{x}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2}$$

$$\vec{x}_2 \cdot \vec{v}_1 = -2, \qquad \vec{v}_1 \cdot \vec{v}_1 = 1^2 + 1^2 + 1^2 = 3$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

Note $\vec{v}_1 \cdot \vec{v}_2 = 1\left(\frac{2}{3}\right) + 1\left(\frac{-1}{3}\right) + 1\left(\frac{-1}{3}\right) = 0$ ✓

The new orthogonal basis is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\}$$

# Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace $W$ of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right) \mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right) \mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1} \left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right) \mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for $W$. Moreover, for each $k = 1, \ldots, p$

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \text{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

# Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for

$$\text{Col } A \text{ where } A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}.$$

These columns are lin. independent (you can check).

Our starting basis is

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

We'll use Gram-Schmidt to get an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{x}_2 \cdot \vec{v}_1 = -36$$

$$\vec{v}_1 \cdot \vec{v}_1 = 12$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1$$

$$= \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \vec{v}_2$$

$$\vec{x}_3 \cdot \vec{v}_1 = 6, \quad \vec{x}_3 \cdot \vec{v}_2 = 30 \qquad \vec{v}_2 \cdot \vec{v}_2 = 12$$

$$\vec{v}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

Our orthogonal basis is

$$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

To get an orthonormal basis, we normalize.

Note that
$$\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = \sqrt{12}$$

Calling the orthonormal basis $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$

we have
$$\vec{w}_1 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1\\ 3\\ 1\\ 1\end{bmatrix}, \quad \vec{w}_2 = \frac{1}{\sqrt{12}}\begin{bmatrix} 3\\ 1\\ 1\\ -1\end{bmatrix}, \quad \text{and}$$

$$\vec{w}_3 = \frac{1}{\sqrt{12}}\begin{bmatrix} -1\\ -1\\ 3\\ -1\end{bmatrix}.$$

# Section 5.1: Eigenvectors and Eigenvalues[1]

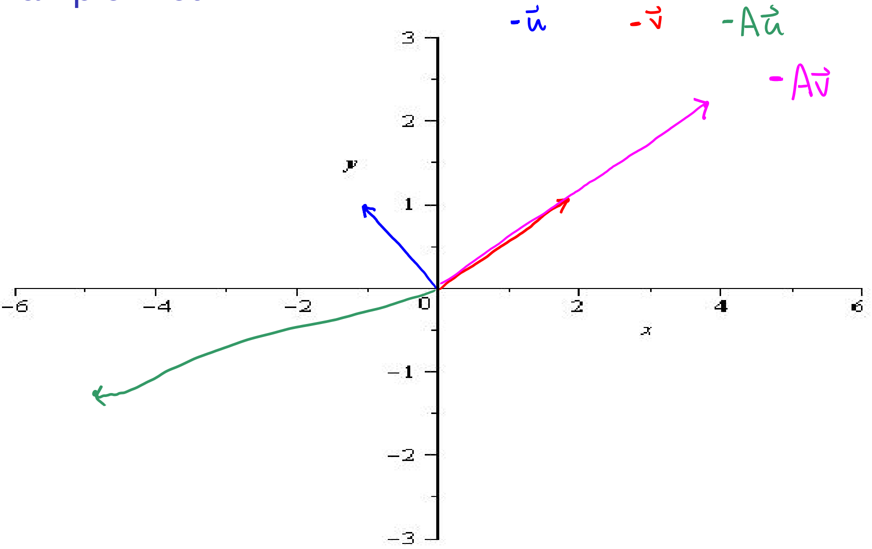
Consider the matrix $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$ and the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Plot $\mathbf{u}$, $A\mathbf{u}$, $\mathbf{v}$, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

$$A\vec{v} = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

[1] We'll forgo section 6.7 and come back if time allows

# Example Plot



# Eigenvalues and Eigenvectors

Note that in this example, the matrix $A$ seems to both stretch and rotate the vector $\mathbf{u}$. But the *action of $A$* on the vector $\mathbf{v}$ is just a stretch/compress.

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant $\lambda$—and for nonzero vector $\mathbf{x}$.

# Definition of Eigenvector and Eigenvalue

**Definition:** Let $A$ be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar $\lambda$ is called an **eigenvector** of the matrix $A$.

A scalar $\lambda$ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix $A$. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* $\lambda$.

Note that built right into this definition is that the eigenvector **$\mathbf{x}$ must be** nonzero!

# Example

The number $\lambda = -4$ is an eigenvalue of the matrix matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

Any eigenvector $\vec{x}$ would have to satisfy $A\vec{x} = -4\vec{x}$.

Letting $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A\vec{x} = -4\vec{x} \Rightarrow \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{array}{l} x_1 + 6x_2 = -4x_1 \\ 5x_1 + 2x_2 = -4x_2 \end{array}$$

This gives a homogeneous system

$$(1-(-4))x_1 + 6x_2 = 0$$

$$5x_1 + (2-(-4))x_2 = 0$$

i.e.

$5x_1 + 6x_2 = 0$ Using rref

$5x_1 + 6x_2 = 0$

$$\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 6/5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{-6}{5} x_2$

$x_2$ - free

So all eigen vectors $\vec{x}$ have to look like

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix}$$

For eigenvectors, we insist

$x_2 \neq 0$.

# Eigenspace

**Definition:** Let $A$ be an $n \times n$ matrix and $\lambda$ and eigenvalue of $A$. The set of all eigenvectors corresponding to $\lambda$ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of $A$ corresponding to $\lambda$.**

**Remark:** The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

## Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of $A$ corresponding to $\lambda$.

Let an eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, then

$A\vec{x} = 2\vec{x}$

$$4x_1 - x_2 + 6x_3 = 2x_1$$
$$2x_1 + x_2 + 6x_3 = 2x_2$$
$$2x_1 - x_2 + 8x_3 = 2x_3$$

$$(4-2)x_1 - x_2 + 6x_3 = 0$$
$$2x_1 + (1-2)x_2 + 6x_3 = 0$$
$$2x_1 - x_2 + (8-2)x_3 = 0$$

New coef. matrix

$$\begin{bmatrix} 4-\lambda & -1 & 6 \\ 2 & 1-\lambda & 6 \\ 2 & -1 & 8-\lambda \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \\ 2 & -1 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & -1/2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = \frac{1}{2}x_2 - 3x_3$$

$$x_2, x_3 - \text{free}$$

$$\vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} \right\}$