April 10 Math 3260 sec. 56 Spring 2018

Section 6.3: Orthogonal Projections

Theorem: If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis of a subspace W of $\mathbb{R}^n$, and $\mathbf{y}$ is any vector in $\mathbb{R}^n$ then

$$\text{proj}_W\, \mathbf{y} = \sum_{j=1}^{p} \left(\mathbf{y} \cdot \mathbf{u}_j\right) \mathbf{u}_j.$$

And, if U is the matrix $U = [\mathbf{u}_1 \quad \cdots \quad \mathbf{u}_p]$, then the above is equivalent to

$$\text{proj}_W\, \mathbf{y} = UU^T\mathbf{y}.$$

Remark: Note that the projection doesn't depend on which basis used as long as it is an orthonormal basis. But this does raise a question about how one might obtain an orthonormal basis.

Section 6.4: Gram-Schmidt Orthogonalization

Question: Given any-old basis for a subspace W of $\mathbb{R}^n$, can we construct an orthogonal basis for that same space?

Example: Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\} = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\-1\end{bmatrix}\right\}$. Find an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ that spans W.

We need $\text{span}\{\vec{v}_1, \vec{v}_2\} = W$. We also need $\vec{v}_1 \cdot \vec{v}_2 = 0$.

Set $\vec{v}_1 = c_1\vec{x}_1 + c_2\vec{x}_2$ and $\vec{v}_2 = d_1\vec{x}_1 + d_2\vec{x}_2$

Let's set $c_1 = 1$, $c_2 = 0$ so $\vec{v}_1 = \vec{x}_1$.

To make sure $\vec{x}_2$ is represented, let's set $d_2 = 1$

So $\vec{v}_2 = \vec{x}_2 + d_1\vec{x}_1 = \vec{x}_2 + d_1\vec{v}_1$.

We insist that $\vec{v}_1 \cdot \vec{v}_2 = 0$. Hence

$$\vec{v}_1 \cdot \vec{v}_2 = (\vec{x}_2 + d_1\vec{v}_1)\cdot\vec{v}_1$$

$$0 = \vec{x}_2\cdot\vec{v}_1 + d_1\vec{v}_1\cdot\vec{v}_1$$ solving for d_1

$$d_1 = \frac{-\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}$$

So $$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\vec{v}_1$$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}$ so $\vec{x}_2\cdot\vec{v}_1 = -2$

$\vec{v}_1\cdot\vec{v}_1 = 3$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix}$$

Note $\vec{v}_1 \cdot \vec{v}_2 = 1\left(\frac{2}{3}\right) + 1\left(\frac{-1}{3}\right) + 1\left(\frac{-1}{3}\right) = 0$

Our orthogonal basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix} \right\}$$

Theorem: Gram Schmidt Process

Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_p\}$ be any basis for the nonzero subspace W of $\mathbb{R}^n$. Define the set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ via

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 \\
\mathbf{v}_2 &= \mathbf{x}_2 - \left(\frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 \\
\mathbf{v}_3 &= \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\right)\mathbf{v}_1 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\right)\mathbf{v}_2 \\
&\vdots \\
\mathbf{v}_p &= \mathbf{x}_p - \sum_{j=1}^{p-1}\left(\frac{\mathbf{x}_p \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}\right)\mathbf{v}_j.
\end{aligned}$$

Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is an orthogonal basis for W. Moreover, for each $k = 1, \ldots, p$

$$\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \mathrm{Span}\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}.$$

Example

Find an orthonormal (that's *orthonormal* not just orthogonal) basis for Col A where $A = \begin{bmatrix} -1 & 6 & 6 \\ 3 & -8 & 3 \\ 1 & -2 & 6 \\ 1 & -4 & -3 \end{bmatrix}$.

The columns are lin. independent (you can check).

Our starting basis is these columns.

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix}$$

Calling the orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$

$$\vec{v}_1 = \vec{x}_1 = \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{V}_2 = \vec{X}_2 - \frac{\vec{X}_2 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1} \vec{V}_1 \qquad \vec{X}_2 \cdot \vec{V}_1 = -36$$

$$\vec{V}_1 \cdot \vec{V}_1 = 12$$

$$\vec{V}_2 = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} - \frac{-36}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ -2 \\ -4 \end{bmatrix} + 3 \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{V}_3 = \vec{X}_3 - \frac{\vec{X}_3 \cdot \vec{V}_1}{\vec{V}_1 \cdot \vec{V}_1} \vec{V}_1 - \frac{\vec{X}_3 \cdot \vec{V}_2}{\vec{V}_2 \cdot \vec{V}_2} \vec{V}_2$$

$$\vec{V}_2 \cdot \vec{V}_2 = 12, \quad \vec{X}_3 \cdot \vec{V}_1 = 6, \quad \vec{X}_3 \cdot \vec{V}_2 = 30$$

$$\vec{V}_3 = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{6}{12} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{30}{12} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ 6 \\ -3 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix}$$

The orthogonal basis is

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

Note $\vec{v}_1 \cdot \vec{v}_3 = 0$ $\quad$ $\vec{v}_2 \cdot \vec{v}_3 = 0$

And $\|\vec{v}_1\| = \|\vec{v}_2\| = \|\vec{v}_3\| = \sqrt{12}$

We can normalize to get the orthonormal basis

$$\left\{ \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{12}}\begin{bmatrix} -1 \\ -1 \\ 3 \\ -1 \end{bmatrix} \right\}$$

Section 5.1: Eigenvectors and Eigenvalues[1]

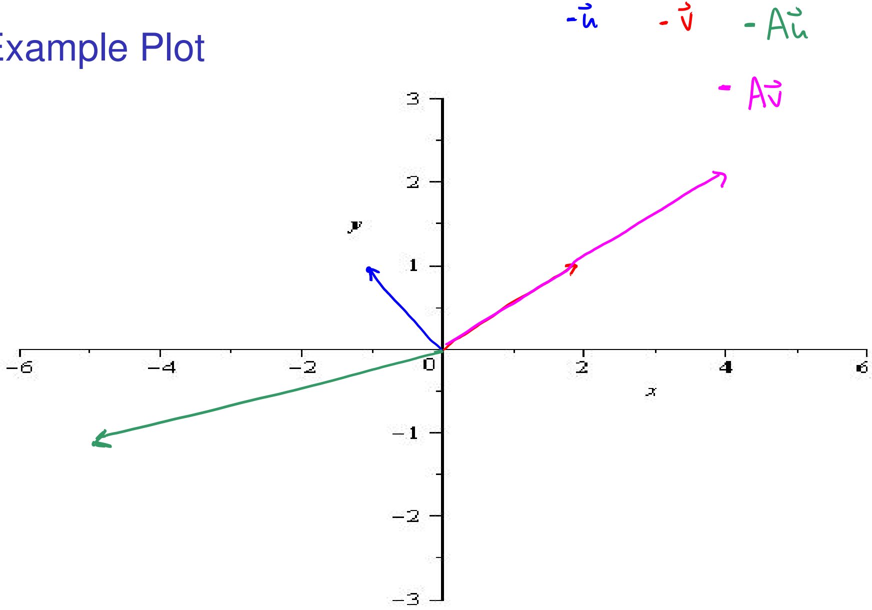
Consider the matrix $A=\begin{bmatrix} 3 & -2\\ 1 & 0\end{bmatrix}$ and the vectors $\mathbf{u}=\begin{bmatrix} -1\\ 1\end{bmatrix}$ and $\mathbf{v}=\begin{bmatrix} 2\\ 1\end{bmatrix}$. Plot **u**, $A\mathbf{u}$, **v**, and $A\mathbf{v}$ on the axis on the next slide.

$$A\vec{u}=\begin{bmatrix} 3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix} -1\\ 1\end{bmatrix}=\begin{bmatrix} -5\\ -1\end{bmatrix}$$

$$A\vec{v}=\begin{bmatrix} 3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix} 2\\ 1\end{bmatrix}=\begin{bmatrix} 4\\ 2\end{bmatrix}$$

[1] We'll forgo section 6.7 and come back if time allows

Example Plot



Eigenvalues and Eigenvectors

Note that in this example, the matrix A seems to both stretch and rotate the vector $\mathbf{u}$. But the *action of A* on the vector $\mathbf{v}$ is just a stretch/compress.

We wish to consider matrices with vectors that satisfy relationships such as

$$A\mathbf{x} = 2\mathbf{x}, \quad \text{or} \quad A\mathbf{x} = -4\mathbf{x}, \quad \text{or more generally} \quad A\mathbf{x} = \lambda\mathbf{x}$$

for constant λ—and for nonzero vector $\mathbf{x}$.

Definition of Eigenvector and Eigenvalue

Definition: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Note that built right into this definition is that the eigenvector **x must be** nonzero!

Example

The number $\lambda = -4$ is an eigenvalue of the matrix matrix $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$. Find the corresponding eigenvectors.

Let eigenvector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. We require $A\vec{x} = -4\vec{x}$.

$$\begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -4\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 6x_2 &= -4x_1 \\ 5x_1 + 2x_2 &= -4x_2 \end{aligned}$$

$$(1-(-4))x_1 + 6x_2 = 0$$

$$5x_1 + (2-(-4))x_2 = 0$$

The new coef. matrix for this homogeneous system is

$$\begin{bmatrix} 5 & 6 \\ 5 & 6 \end{bmatrix}$$

We can use row reduction $\begin{bmatrix} 5 & 6 & 0 \\ 5 & 6 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & \frac{6}{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = \frac{-6}{5} x_2$

x_2 - free

The eigenvectors are of the form

$$\vec{x} = x_2 \begin{bmatrix} -6/5 \\ 1 \end{bmatrix} \quad \text{for } x_2 \neq 0.$$

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.