April 13 Math 1190 sec. 62 Spring 2017

Section 5.3: The Fundamental Theorem of Calculus

Theorem: FTC (The Fundamental Theorem of Calculus part 1)
If f is continuous on $[a, b]$ and the function g is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then g is continuous on $[a, b]$ and differentiable on (a, b). Moreover

$$g'(x) = f(x).$$

This means that the new function g is an **antiderivative** of f on (a, b)!
"FTC" = "fundamental theorem of calculus"

Theorem: The Fundamental Theorem of Calculus (part 2)

If f is continuous on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f on $[a,b]$. (i.e. $F'(x) = f(x)$)

Example

Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(t)\,dt = \tan t \Big|_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$

$= \sqrt{3} - 1$

Recall $\frac{d}{dt}\tan t = \sec^2 t$

Example

Evaluate $\displaystyle \frac{d}{dx}\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt$ in two ways.

(a) Using the FTC part 1.

FTC says $\displaystyle \frac{d}{dx}\int_a^x f(t)\,dt = f(x)$

$$\frac{d}{dx}\int_{\pi/4}^{x} \sec^2(t)\,dt = \sec^2(x)$$

here $f(t) = \sec^2(t)$.

Example

Evaluate $\dfrac{d}{dx}\displaystyle\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt$ in two ways.

(b) By using the FTC part 2 to evaluate $g(x) = \displaystyle\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt,$

then using regular derivative rules to find $g'(x)$.

First treat "x" as "b".

$$g(x) = \int_{\pi/4}^{x} \sec^2 t\, dt = \tan t \Big|_{\pi/4}^{x} = \tan x - \tan\frac{\pi}{4} = \tan x - 1$$

Now find $g'(x)$:

$$g(x) = \tan x - 1 \Rightarrow g'(x) = \sec^2 x - 0 \Rightarrow g'(x) = \sec^2 x$$

Question

$$\int_1^9 \frac{1}{2}u^{-1/2}\,du = u^{1/2}\Big|_1^9 = \sqrt{u}\Big|_1^9 = \sqrt{9} - \sqrt{1} = 2$$

(a) 8

$$\frac{1}{2}u^{-1/2} \longrightarrow \frac{1}{2}\frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} = \frac{1}{2}\frac{u^{1/2}}{1/2} = u^{1/2}$$

(b) $\frac{13}{54}$

(c) 2

(d) $-\frac{1}{3}$

Question

(remember: $\int_b^a f(z)\,dz = -\int_a^b f(z)\,dz$)

$$\frac{d}{dx}\int_x^3 \tan(z)\,dz \quad = \quad \frac{d}{dx}\; -\int_3^x \tan(z)\,dz \;=\; -\tan x$$

(a) $\tan(x)$

(b) $\ln|\sec(3)| - \ln|\sec x|$

(c) $-\tan(x)$

(d) $\sec^2(x)$

Caveat! The FTC doesn't apply if f is not continuous!

The function $f(x) = \frac{1}{x^2}$ is positive everywhere on its domain. Now consider the calculation

$$\int_{-1}^{2} \frac{1}{x^2}\,dx = \left.\frac{x^{-1}}{-1}\right|_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$$

Is this believable? Why or why not?

No, f is never negative, $-\frac{3}{2}$ can't be right!

An Observation

If f is differentiable on $[a, b]$, note that

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

This says that:

The integral of the **rate of change** of f over the interval $[a, b]$ is the **net change** of the function, $f(b) - f(a)$, over this interval.

Remember the example: the *area* under the velocity curve gave the net change in position!

Section 5.4: Properties of the Definite Integral

Suppose that f and g are integable on $[a, b]$ and let k be constant.

I. $$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

II. $$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

II. $$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Examples

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

(i) $\displaystyle\int_1^4 -2f(x)\,dx$ $= -2\int_1^4 f(x)\,dx$ prop. I.

$= -2(3) = -6$

(ii) $\displaystyle\int_1^4 [f(x)+3g(x)]\,dx$ $= \int_1^4 f(x)\,dx + \int_1^4 3g(x)\,dx$ prop. II

$= \int_1^4 f(x)\,dx + 3\int_1^4 g(x)\,dx$ prop I

$= 3 + 3(-7) = 3 - 21 = -18$

Question

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

$$\int_1^4 [g(x) - 3f(x)]\,dx \;=\; \int_1^4 g(x)\,dx - 3\int_1^4 f(x)\,dx$$

$$= -7 - 3(3) = -7 - 9 = -16$$

(a) 16

(b) -16

(c) -2

(d) 2

The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a, b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\,dx =$$

$$k_1 \int_a^b f_1(x)\,dx + k_2 \int_a^b f_2(x)\,dx + \cdots + k_n \int_a^b f_n(x)\,dx$$

Example

Evaluate $\displaystyle\int_1^2 \frac{x^3+2x^2+4}{x}\,dx$ $= \displaystyle\int_1^2 \left(\frac{x^3}{x} + \frac{2x^2}{x} + \frac{4}{x}\right) dx$

$= \displaystyle\int_1^2 \left(x^2 + 2x + \frac{4}{x}\right) dx = \int_1^2 x^2\,dx + \int_1^2 2x\,dx + \int_1^2 4\left(\frac{1}{x}\right) dx$

$= \displaystyle\int_1^2 x^2\,dx + 2\int_1^2 x\,dx + 4\int_1^2 \frac{1}{x}\,dx$

$= \displaystyle\frac{x^3}{3}\Big|_1^2 + \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_1^2 + 4\ln|x|\,\Big|_1^2$

$= \displaystyle\frac{2^3}{3} - \frac{1^3}{3} + \left(2^2 - 1^2\right) + \left(4\ln|2| - 4\ln|1|\right)$

$F(x) = \dfrac{x^3}{3}$ $\quad F(2) - F(1)$

$$= \frac{8}{3} - \frac{1}{3} + 4 - 1 + 4\ln 2 - 0$$

$$= \frac{7}{3} + 3 + 4\ln 2 \qquad = \frac{7}{3} + \frac{9}{3} + 4\ln 2$$

$$= \frac{16}{3} + 4\ln 2$$

We can write

$$\int_1^2 \left(x^2 + 2x + \frac{4}{x}\right)dx = \frac{x^3}{3} + x^2 + 4\ln|x| \Big|_1^2$$

$$= \frac{2^3}{3} + 2^2 + 4\ln|2| - \left(\frac{1^3}{3} + 1^2 + 4\ln|1|\right)$$

Properties of Definite Integrals Continued...

If f is integrable on any interval containing the numbers a, b, and c, then

(IV) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

e.g. $$\int_2^7 f(x)dx = \int_2^5 f(x)dx + \int_5^7 f(x)dx$$

or $$\int_2^5 f(x)dx = \int_2^7 f(x)dx + \int_7^5 f(x)dx$$

Properties: Bounds on Integrals

(V) If $f(x) \le g(x)$ for $a \le x \le b$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$

(VI) And, as an immediate consequence of (V), if $m \le f(x) \le M$ for $a \le x \le b$, then

$$\underbrace{m(b-a)}_{\int_a^b m\,dx} \le \int_a^b f(x)\,dx \le \underbrace{M(b-a)}_{\int_a^b M\,dx}.$$

If f is continuous on $[a, b]$, we can take m to be the absolute minimum value and M the absolute maximum value of f as guaranteed by the Extreme Value Theorem.

Average Value of a Function and the Mean Value Theorem

Defintion: Let f be continuous on the closed interval $[a,b]$. Then the average value of f on $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem: (The Mean Value Theorem for Integrals) If f is continuous on the interval $[a,b]$, then there exists a number u in $[a,b]$ such that

$$f(u) = f_{avg}, \quad \text{i.e.} \quad \int_a^b f(x)\,dx = f(u)(b-a).$$

Question

Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, compute

$$f_{avg} = \frac{1}{4-0}\int_0^4 x^{1/2}\,dx$$

(a) $\frac{16}{3}$

(b) $\frac{4}{3}$

(c) $\frac{1}{2}$

(d) 2

$$= \frac{1}{4}\left(\frac{x^{3/2}}{3/2}\Bigg|_0^4\right.$$

$$4^{3/2} = 2^3 = 8$$

$$= \frac{1}{4}\left(\frac{2}{3}x^{3/2}\Bigg|_0^4 = \frac{1}{4}\left(\frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}\right)\right.$$

$$= \frac{1}{4}\left(\frac{2}{3}\cdot 8\right) = \frac{1}{4}\cdot\frac{16}{3} = \frac{4}{3}$$

Question

Find the value of f guaranteed by the MVT for integrals for $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, find u such that

$$f(u) = f_{avg} = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{4}{3}$$

$f(u) = \frac{4}{3} \Rightarrow \sqrt{u} = \frac{4}{3}$

$(\sqrt{u})^2 = \left(\frac{4}{3}\right)^2$

$u = \frac{16}{9}$

(a) $\sqrt{\frac{4}{3}}$

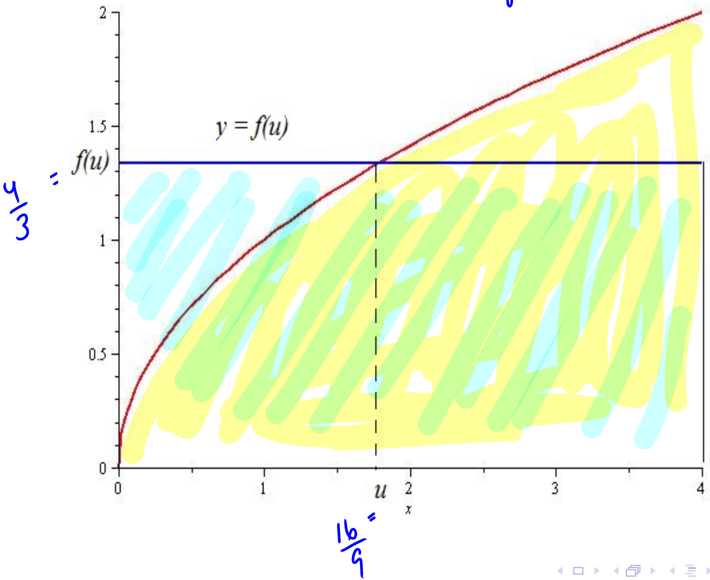
(b) $\frac{2}{\sqrt{3}}$

(c) $\frac{16}{9}$

(d) $\frac{16}{3}$

MVT for Integrals Example

MVT says the yellow area = blue area



Evaluate Each Integral

(a) $\int_2^1 (t+1)^2\,dt$

Expand square first

$(t+1)^2 = t^2 + 2t + 1$

$$= \int_2^1 \left(t^2 + 2t + 1\right)\,dt$$

$$= \frac{t^3}{3} + \cancel{2}\,\frac{t^2}{\cancel{2}} + t \Big|_2^1 = \left(\frac{1^3}{3} + 1^2 + 1\right) - \left(\frac{2^3}{3} + 2^2 + 2\right)$$

$$= \frac{1}{3} + 2 - \frac{8}{3} - 6 = \frac{-7}{3} - 4 = \frac{-7}{3} - \frac{12}{3} = \frac{-19}{3}$$

Question

(b) $\displaystyle\int_1^3 x(3x+2)\,dx$ $= \displaystyle\int_1^3 (3x^2+2x)\,dx$

$= x^3 + x^2 \Big|_1^3 = 34$

(a) 86

(b) 34

(c) 47

(d) 28

(c) $\displaystyle\int_0^{\pi/4} \tan^2\theta \, d\theta$

Recall

$\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \tan^2\theta = \sec^2\theta - 1$

$$= \int_0^{\pi/4} (\sec^2\theta - 1) \, d\theta$$

$$= \tan\theta - \theta \Big|_0^{\pi/4} = \tan\frac{\pi}{4} - \frac{\pi}{4} - (\tan 0 - 0)$$

$$= 1 - \frac{\pi}{4}$$

(d) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x}$ $=$ $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/2} \csc^2 x\,dx$

$= -\cot x \Big|_{\pi/4}^{\pi/2}$

$= -\cot\frac{\pi}{2} - \left(-\cot\frac{\pi}{4}\right)$

$= 0 - (-1) = 1$

(a) 1

(b) -1

(c) $\dfrac{1}{1-\frac{1}{\sqrt{2}}}$

(d) It is undefined since $\cos(\pi/2) = 0$.