# April 13 Math 1190 sec. 63 Spring 2017

## Section 5.3: The Fundamental Theorem of Calculus

**Theorem: FTC** (The Fundamental Theorem of Calculus part 1)
If $f$ is continuous on $[a,b]$ and the function $g$ is defined by

$$g(x) = \int_a^x f(t)\,dt \quad \text{for} \quad a \le x \le b,$$

then $g$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Moreover

$$g'(x) = f(x).$$

This means that the new function $g$ is an **antiderivative** of $f$ on $(a,b)$!
"FTC" = "fundamental theorem of calculus"

# Theorem: The Fundamental Theorem of Calculus (part 2)

If $f$ is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where $F$ is any antiderivative of $f$ on $[a, b]$. (i.e. $F'(x) = f(x)$)

# Example

Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sec^2(t)\,dt = \tan t \Big|_{\pi/4}^{\pi/3} = \tan\frac{\pi}{3} - \tan\frac{\pi}{4}$

$= \sqrt{3} - 1$

recall $\frac{d}{dt}\tan t = \sec^2 t$

# Example

Evaluate $\dfrac{d}{dx}\displaystyle\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt$ in two ways.

(a) Using the FTC part 1.

FTC says $\dfrac{d}{dx}\displaystyle\int_{a}^{x} f(t)\,dt = f(x)$

$$\frac{d}{dx}\int_{\pi/4}^{x} \sec^2 t\,dt = \sec^2 x$$

here $f(t) = \sec^2 t$

# Example

Evaluate $\quad \dfrac{d}{dx}\displaystyle\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt \quad$ in two ways.

(b) By using the FTC part 2 to evaluate $\quad g(x) = \displaystyle\int_{\frac{\pi}{4}}^{x} \sec^2(t)\,dt,$

then using regular derivative rules to find $g'(x)$.

First, treat "x" like "b".

$$\int_{\pi/4}^{x} \sec^2(t)\,dt = \tan t \Big|_{\pi/4}^{x} = \tan x - \tan \pi/4 = \tan x - 1$$

Now, find $g'(x)$.

$$g(x) = \tan x - 1 \Rightarrow g'(x) = \sec^2(x) - 0 = \sec^2(x)$$

# Question

$$\int_1^2 \frac{1}{1+x^2}\,dx \quad = \tan^{-1} x \Big|_1^2 = \tan^{-1} 2 - \tan^{-1} 1$$

(a) $-\frac{1}{6}$

(b) $-\frac{4}{5}$

(c) $\tan^{-1}(2)$

(d) $\tan^{-1}(2) - \tan^{-1}(1)$

# Question

(remember: $\int_b^a f(z)\,dz = -\int_a^b f(z)\,dz$)

$$\frac{d}{dx}\int_x^3 \tan(z)\,dz = \frac{d}{dx}\left(-\int_3^x \tan(z)\,dz\right)$$

$$= -\frac{d}{dx}\int_3^x \tan(z)\,dz$$

$$= -\tan(x)$$

(a) $\tan(x)$

(b) $\ln|\sec(3)| - \ln|\sec x|$

(c) $-\tan(x)$

(d) $\sec^2(x)$

# Section 5.4: Properties of the Definite Integral

Suppose that $f$ and $g$ are integable on $[a, b]$ and let $k$ be constant.

I. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$

II. $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

II. $\displaystyle\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

# Examples

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

(i) $\displaystyle\int_1^4 -2f(x)\,dx$ $= -2\displaystyle\int_1^4 f(x)dx$ prop. I.

$= -2(3) = -6$

(ii) $\displaystyle\int_1^4 [f(x)+3g(x)]\,dx$ $= \displaystyle\int_1^4 f(x)dx + \int_1^4 3g(x)dx$ prop II

$= \displaystyle\int_1^4 f(x)dx + 3\int_1^4 g(x)dx$ prop I

$= 3 + 3(-7) = 3-21 = -18$

# Question

Suppose $\int_1^4 f(x)\,dx = 3$ and $\int_1^4 g(x)\,dx = -7$. Evaluate

$$\int_1^4 [g(x) - 3f(x)]\,dx \quad = \int_1^4 g(x)\,dx - 3\int_1^4 f(x)\,dx$$

$$= -7 - 3(3) = -16$$

(a) 16

(b) $-16$

(c) $-2$

(d) 2

# The Sum/Difference in General

If $f_1, f_2, \ldots, f_n$ are integrable on $[a,b]$ and $k_1, k_2, \ldots, k_n$ are constants, then

$$\int_a^b [k_1 f_1(x) + k_2 f_2(x) + \cdots + k_n f_n(x)]\, dx =$$

$$k_1 \int_a^b f_1(x)\, dx + k_2 \int_a^b f_2(x)\, dx + \cdots + k_n \int_a^b f_n(x)\, dx$$

# Example

Evaluate $\displaystyle\int_1^2 \frac{x^3+2x^2+4}{x}\,dx = \int_1^2 \left(\frac{x^3}{x} + \frac{2x^2}{x} + \frac{4}{x}\right) dx$

$$= \int_1^2 \left(x^2 + 2x + 4\cdot\frac{1}{x}\right) dx$$

$$= \int_1^2 x^2\,dx + \int_1^2 2x\,dx + \int_1^2 4\cdot\frac{1}{x}\,dx$$

$$= \int_1^2 x^2\,dx + 2\int_1^2 x\,dx + 4\int_1^2 \frac{1}{x}\,dx$$

$$= \frac{x^3}{3}\Big|_1^2 + \cancel{2}\,\frac{x^2}{\cancel{2}}\Big|_1^2 + 4\ln|x|\,\Big|_1^2$$

$$= \frac{2^3}{3} - \frac{1^3}{3} + \left(2^2 - 1^2\right) + \left(4\ln|2| - 4\ln|1|\right)$$

$$= \frac{8}{3} - \frac{1}{3} + 4 - 1 + 4\ln 2 - 0$$

$$= \frac{7}{3} + 3 + 4\ln 2 = \frac{7}{3} + \frac{9}{3} + 4\ln 2 = \frac{16}{3} + 4\ln 2$$

we could write

$$\int_1^2 \left(x^2 + 2x + 4\cdot\frac{1}{x}\right)dx = \frac{x^3}{3} + x^2 + 4\ln|x| \Big|_1^2$$

$$= \frac{2^3}{3} + 2^2 + 4\ln|2| - \left(\frac{1^3}{3} + 1^2 + 4\ln|1|\right)$$

# Properties of Definite Integrals Continued...

If $f$ is integrable on any interval containing the numbers $a$, $b$, and $c$, then

(IV) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

e.g. $$\int_2^9 f(x)\,dx = \int_2^7 f(x)\,dx + \int_7^9 f(x)\,dx$$

also $$\int_2^7 f(x)\,dx = \int_2^9 f(x)\,dx + \int_9^7 f(x)\,dx$$

# Properties: Bounds on Integrals

(V) If $f(x) \leq g(x)$ for $a \leq x \leq b$, then $\displaystyle\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

(VI) And, as an immediate consequence of (V), if $m \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$\underbrace{m(b-a)}_{\int_a^b m\,dx} \leq \int_a^b f(x)\,dx \leq \underbrace{M(b-a)}_{\int_a^b M\,dx}.$$

If $f$ is continuous on $[a, b]$, we can take $m$ to be the absolute minimum value and $M$ the absolute maximum value of $f$ as guaranteed by the Extreme Value Theorem.

# Average Value of a Function and the Mean Value Theorem

**Defintion:** Let $f$ be continuous on the closed interval $[a,b]$. Then the average value of $f$ on $[a,b]$ is

$$f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

**Theorem:** (The Mean Value Theorem for Integrals) If $f$ is continuous on the interval $[a,b]$, then there exists a number $u$ in $[a,b]$ such that

$$f(u) = f_{avg}, \quad \text{i.e.} \quad \int_a^b f(x)\,dx = f(u)(b-a).$$

# Question

Find the average value of $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, compute

$$f_{avg} = \frac{1}{4-0}\int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{1}{4}\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^4$$

$$= \frac{1}{4}\left(\frac{2}{3}x^{3/2}\right|_0^4 = \frac{1}{4}\left(\frac{2}{3}(4)^{3/2} - \frac{2}{3}(0)^{3/2}\right)$$

$$= \frac{1}{4}\left(\frac{2}{3}\cdot 8\right) = \frac{1}{4}\,\frac{16}{3} = \frac{4}{3}$$

(a) $\frac{16}{3}$

(b) $\frac{4}{3}$

(c) $\frac{1}{2}$

(d) 2

# Question

Find the value of $f$ guaranteed by the MVT for integrals for $f(x) = \sqrt{x}$ on the interval $[0, 4]$. That is, find $u$ such that

$$f(u) = f_{avg} = \frac{1}{4}\int_0^4 x^{1/2}\,dx = \frac{4}{3}$$

$f(u) = \frac{4}{3} \Rightarrow \sqrt{u} = \frac{4}{3}$

$(\sqrt{u})^2 = \left(\frac{4}{3}\right)^2$

$u = \frac{16}{9}$

(a) $\sqrt{\frac{4}{3}}$

(b) $\frac{2}{\sqrt{3}}$

(c) $\frac{16}{9}$

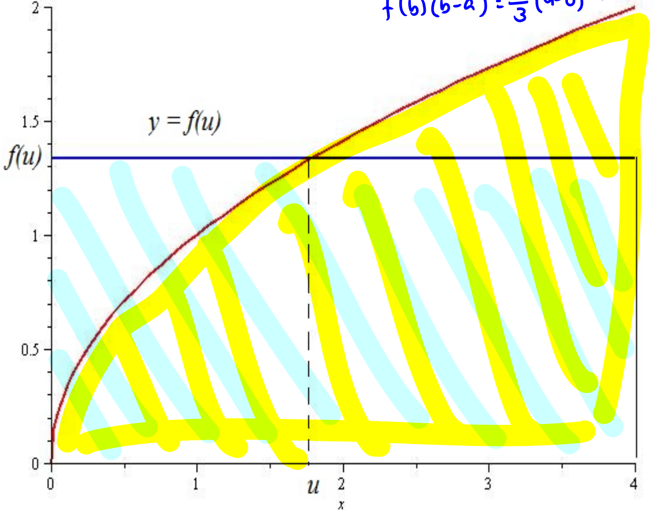
(d) $\frac{16}{3}$

# MVT for Integrals Example

$\int_0^4 \sqrt{x}\,dx$ = yellow area

$f(b)(b-a) = \frac{4}{3}(4-0)$ = blue area

$\frac{4}{3}$ =




$\frac{16}{9}$

# Evaluate Each Integral

(a) $\int_2^1 (t+1)^2\,dt$

Square $t+1$ first

$(t+1)^2 = t^2+2t+1$

$= \int_2^1 (t^2+2t+1)\,dt$

$= \frac{t^3}{3} + \cancel{2}\frac{t^2}{\cancel{2}} + t \Big|_2^1$

$= \frac{1^3}{3} + 1^2 + 1 - \left(\frac{2^3}{3} + 2^2 + 2\right) = \frac{1}{3} + 2 - \left(\frac{8}{3} + 6\right)$

$= \frac{1}{3} - \frac{8}{3} + 2 - 6 = \frac{-7}{3} - 4 = \frac{-7}{3} - \frac{12}{3} = \frac{-19}{3}$

# Question

(b) $\int_1^3 x(3x+2)\,dx$ $= \int_1^3 (3x^2 + 2x)\,dx$

$= x^3 + x^2 \Big|_1^3 = 34$

(a) 86

(b) 34

(c) 47

(d) 28

(c) $\int_0^{\pi/4} \tan^2\theta \, d\theta$

Recall the Trig ID

$\tan^2\theta + 1 = \sec^2\theta$

$\Rightarrow \tan^2\theta = \sec^2\theta - 1$

$= \int_0^{\pi/4} (\sec^2\theta - 1) \, d\theta$

$= \tan\theta - \theta \Big|_0^{\pi/4} = \tan\pi/4 - \pi/4 - (\tan 0 - 0)$

$= 1 - \frac{\pi}{4} - 0$

$= 1 - \pi/4$

(d) $\displaystyle\int_{\pi/4}^{\pi/2} \frac{dx}{\sin^2 x}$

$$= \int_{\pi/4}^{\pi/2} \frac{1}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/2} \csc^2 x\,dx$$

$$= -\cot x \Big|_{\pi/4}^{\pi/2}$$

$$= -\cot\frac{\pi}{2} - \left(-\cot\frac{\pi}{4}\right)$$

$$= -0 - (-1) = 1$$

(a) 1

(b) $-1$

(c) $\frac{1}{1-\frac{1}{\sqrt{2}}}$

(d) It is undefined since $\cos(\pi/2) = 0$.