April 13 Math 2254H sec 015H Spring 2015

Section 11.9: Functions as Power Series

Motivating Example: Let

$$f(x) = \frac{1}{1-x}, \quad \text{for} \quad -1 < x < 1.$$

Use the well known relation $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$ to express f as a power series.

$\frac{1}{1-x} = \frac{a}{1-r}$ if $a=1$ and $r=x$

for $-1<x<1$, $|r| = |x| < 1$

so

$f(x) = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

Using Part of a Series to Approximate *f*

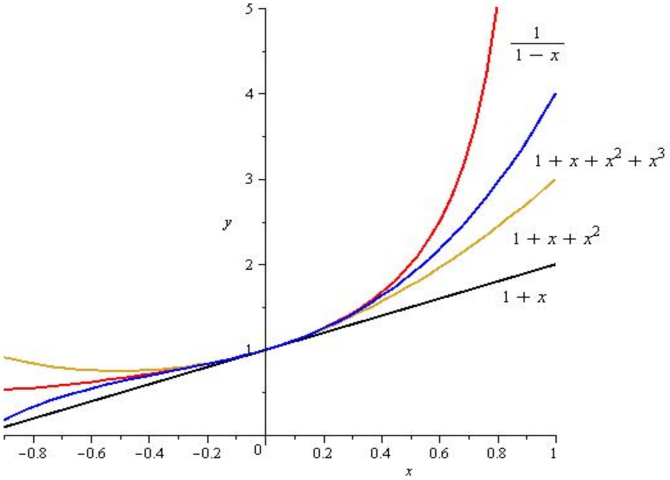

Figure: Plot of *f* along with the first 2, 3, and 4 terms of the series. Near the center, the graphs agree well. The fit breaks down away from the center.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

Find a power series representation, in powers of x, of the rational function. Indicate the interval of convergence.

$$f(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

If $a=1$ and $r=-x^2$

then $\frac{1}{1-(-x^2)} = \frac{a}{1-r}$

$$f(x) = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \dots$$

Convergence requires $|-x^2| < 1$

$\Rightarrow \quad |x^2| < 1 \quad \Rightarrow \quad |x| < 1$

$|x| < 1 \Rightarrow -1 < x < 1$

Since $\sum_{n=0}^{\infty} ar^n$ diverges if $|r| = 1$

our series diverges if $|x| = 1$

So the interval convergence is $(-1, 1)$.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

Find a power series representation, in powers of x, of the rational function. Indicate the interval of convergence.

$$f(x) = \frac{1}{x-3} = \frac{-1}{3-x} = \frac{-1}{3\left(1-\frac{x}{3}\right)} = \frac{-1/3}{1-\frac{x}{3}}$$

This has the right form for $a = \frac{-1}{3}$

and $r = \frac{x}{3}$

$$f(x) = \sum_{n=0}^{\infty} \frac{-1}{3}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{3}\,\frac{x^n}{3^n} = \sum_{n=0}^{\infty} -\frac{x^n}{3^{n+1}}$$

Convergence requires $\left|\frac{x}{3}\right| < 1$ i.e. $|x| < 3$.

$$-3 < x < 3$$

is the interval of convergence.

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \text{ for } |r| < 1$$

Find a power series representation, in powers of x, of the rational function, and choose the index so that the power on x is the index (e.g. n). Indicate the interval of convergence.

$$f(x) = \frac{4x^2}{x-3}$$

Recall $\sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}} = \frac{1}{x-3}$

for $-3 < x < 3$

So for $-3 < x < 3$

$$f(x) = 4x^2 \sum_{n=0}^{\infty} \frac{-x^n}{3^{n+1}} = \sum_{n=0}^{\infty} \frac{-4x^{n+2}}{3^{n+1}}$$

If we set $k = n+2$ then $n+1 = k-1$

and $k = 2$ when $n = 0$.

$$f(x) = \sum_{k=2}^{\infty} \frac{-4\,x^k}{3^{k-1}}$$

An Alternative Power Series

Find a power series representation, in powers of $x+1$, of the rational function. Indicate the interval of convergence.

$$f(x) = \frac{1}{x-3} = \frac{1}{x+1-1-3} = \frac{1}{(x+1)-4}$$

$$= \frac{1}{-4\left[1-\frac{(x+1)}{4}\right]} = \frac{-1/4}{1-\frac{(x+1)}{4}}$$

For a geometric series

$a = -1/4$

$r = \frac{x+1}{4}$

$$f(x) = \sum_{n=0}^{\infty} \frac{-1}{4}\left(\frac{x+1}{4}\right)^n = \sum_{n=0}^{\infty} \frac{-(x+1)^n}{4^{n+1}}$$

Convergence requires $\left|\frac{x+1}{4}\right| < 1 \Rightarrow |x+1| < 4$

$-4 < x+1 < 4 \Rightarrow -5 < x < 3$

Theorem: Differentiation and Integration

Theorem: Let $\sum c_n(x-a)^n$ have positive radius of convergence R, and let the function f be defined by this power series

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n = c_0+c_1(x-a)+c_2(x-a)^2+\cdots$$

Then f is differentiable on $(a-R, a+R)$. Moreover,

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+\cdots=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1}, \quad \text{and}$$

$$\begin{aligned}\int f(x)\,dx &= C+c_0(x-a)+c_1\frac{(x-a)^2}{2}+c_2\frac{(x-a)^3}{3}+\cdots \\ &= C+\sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}\end{aligned}$$

The radius of convergence for each of these series is R.

Guess That Function

Let $f(x)$ be given by the following power series. Take a couple of derivatives, and see if you can guess exactly what function f is.

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$f'(x) = 1 + \frac{2x}{2!} + \frac{3x^2}{3!} + \frac{4x^3}{4!} + \frac{5x^4}{5!} + \cdots$$

$$= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$= f(x) \qquad f'(x) = f(x) \implies f(x) = e^x$$

Finding Power Series Representations

Find a power series representation for $f(x)$, and state the interval of convergence.

$$f(x) = \frac{1}{(x-1)^2}$$

$$\frac{d}{dx}\frac{1}{1-x} = \frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1)$$

$$= (1-x)^{-2}$$

$$= \frac{1}{(1-x)^2}$$

$$f(x) = \frac{1}{(1-x)^2} = \frac{d}{dx}\frac{1}{1-x}$$

$$= \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) \quad \text{for } |x| < 1$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

The interval of convergence is the same as the interval for the original one $(-1, 1)$.

$$\frac{d}{dx}\sum_{n=0}^{\infty} x^n = \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right)$$

$$= 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Finding Power Series Representations

Find a power series representation for $g(x)$, and determine the interval of convergence.

$$g(x) = \tan^{-1} x$$

Note $g'(x) = \dfrac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}$

for $-1 < x < 1$

$$g(x) = \int \frac{1}{1+x^2}\,dx = \int \left(\sum_{n=0}^{\infty} (-1)^n x^{2n} \right) dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$g(x) = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$g(0) = \tan^{-1}(0) = C + 0 - \frac{0^3}{3} + \frac{0^5}{5} - \frac{0^7}{7} + \dots$$

$$0 = C$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| < 1$$

If $x = 1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$

This is a convergent alternating series.

If $x=-1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n (-1)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} \frac{-(-1)^n}{2n+1}$

This is also a convergent alternating series.

So in fact

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad -1 \le x \le 1$$

A series convergent to π

Use the power series for $\tan^{-1} x$ to deduce a series of rational numbers that converges to π.

$$\tan^{-1}(1) = \sum_{n=0}^{\infty} \frac{(-1)^n \, 1^{2n+1}}{2n+1}$$

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

$$\Rightarrow \quad \pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1} = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \dots$$