April 14 Math 2254H sec 015H Spring 2015

Section 11.9: Functions as Power Series

Find a power series representation for $\ln(4+x)$ and show that it converges on $-4 < x \leq 4$.

$$\frac{d}{dx}\ln(4+x) = \frac{1}{4+x} = \frac{1}{4\left(1+\frac{x}{4}\right)} = \frac{1/4}{1-\left(\frac{-x}{4}\right)}$$

for $a = \frac{1}{4}$ $\quad r = \frac{-x}{4}$

$$\frac{d}{dx}\ln(4+x) = \sum_{n=0}^{\infty} \frac{1}{4}\left(\frac{-x}{4}\right)^n$$

for $\left|\frac{-x}{4}\right| < 1$ i.e.

$|x| < 4$

Let $g(x) = \ln(4+x)$

$$\frac{d}{dx}\ln(4+x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{4^{n+1}} \quad \text{for } -4 < x < 4$$

$$\ln(4+x) = \int \frac{d}{dx}\ln(4+x)\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}} \frac{x^{n+1}}{n+1}$$

$$= C + \frac{x}{4} - \frac{x^2}{2\cdot 16} + \frac{x^3}{3\cdot 64} - \ldots$$

$$g(0) = \ln(4) = C + 0 - 0 + \ldots \Rightarrow C = \ln 4$$

$$\ln(4+x) = \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)\, 4^{n+1}} \qquad -4 < x < 4$$

End point check:

$x = 4$ $$\sum_{n=0}^{\infty} \frac{(-1)^n\, 4^{n+1}}{(n+1)\, 4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$$

The alt. harmonic series which converges.

$x = -4$ $$\sum_{n=0}^{\infty} \frac{(-1)^n (-4)^{n+1}}{(n+1)\, 4^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{-4}{4}\right)^{n+1}}{n+1}$$

$$= \sum_{n=0}^{\infty} \frac{-1}{n+1}$$

negative harmonic series which diverges.

So

$$\ln(4+x) = \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{4^{n+1}(n+1)} \quad \text{for } -4 < x \le 4$$

Approximation Using a Power Series

Use the first two nonzero terms of a power series for $\frac{1}{1+x^3}$ to approximate the integral

$$\int_0^{0.1} \frac{dx}{1+x^3}$$

$$\frac{1}{1+x^3} = \frac{1}{1-(-x^3)} = \sum_{n=0}^{\infty} (-x^3)^n$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{3n} \qquad |x| < 1$$

$$= 1 - x^3 + x^6 - \dots$$

$$\int_0^{0.1} \frac{1}{1+x^3}\,dx \approx \int_0^{0.1} (1-x^3)\,dx$$

$$= x - \frac{x^4}{4} \Big|_0^{0.1}$$

$$= 0.1 - \frac{1}{4}(0.1)^4 = 0.1 - 0.25(0.0001)$$

$$= 0.1 - 0.000025$$

$$= 0.099975$$

Exercise

Find a power series representation for $\frac{x}{2x^2+1}$. What is the interval of convergence?

$$f(x) = \frac{x}{2x^2+1} = \frac{x}{1-(-2x^2)}$$

$$= x\sum_{n=0}^{\infty}(-2x^2)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n 2^n x^{2n+1}$$

$$|-2x^2| < 1$$

$$|x|^2 < \frac{1}{2}$$

$$|x| < \frac{1}{\sqrt{2}}$$

The Interval is

$$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Find a power series for $\int \frac{\ln(1+x)}{x}\,dx$ for $0 < x < 1$.

$$\ln(1+x) = \int \frac{dx}{1+x} = C + \int \sum_{n=0}^{\infty} (-1)^n x^n$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1} \qquad \ln(1) = C = 0$$

$$\int \frac{\ln(1+x)}{x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{(n+1)x}\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n+1}\,dx$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{(n+1)^2}$$