## April 17 Math 2306 sec. 57 Spring 2018

## Section 17: Fourier Series: Trigonometric Series

Suppose $f(x)$ is defined for $-\pi<x<\pi$. We would like to know how to write $f$ as a series in terms of sines and cosines.

Task: Find coefficients (numbers) $a_{0}, a_{1}, a_{2}, \ldots$ and $b_{1}, b_{2}, \ldots$ such that ${ }^{1}$

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \sin n x\right)
$$

[^0]
## The Fourier Series of $f(x)$ on $(-\pi, \pi)$

The Fourier series of the function $f$ defined on $(-\pi, \pi)$ is given by

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos n x+b_{n} \sin n x\right) .
$$

Where

$$
\begin{aligned}
a_{0} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) d x \\
a_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos n x d x, \quad \text { and } \\
b_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin n x d x
\end{aligned}
$$

## Some convenient observations

For every integer $n$

$$
\sin (n \pi)=0
$$

For integers $n$

$$
\begin{aligned}
\cos (n \pi) & =\cos (-n \pi)=(-1)^{n} \\
& :\left\{\begin{array}{l}
1, n-\text { esen } \\
-1, n-\text { odd }
\end{array}\right.
\end{aligned}
$$

Example
Find the Fourier series of the piecewise defined function

$$
\begin{gathered}
f(x)= \begin{cases}0, & -\pi<x<0 \\
x, & 0 \leq x<\pi\end{cases} \\
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty} a_{n} \cos (n x)+b_{n} \sin (n x) \xrightarrow[-\pi]{\pi} \\
a_{0}=\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) d x \\
\\
=\frac{1}{\pi}\left(\int_{-\pi}^{0} 0 d x+\int_{0}^{\pi} x d x\right)=\frac{1}{\pi}\left[\left.\frac{x^{2}}{2}\right|_{0} ^{\pi}=\frac{\pi^{2}}{2 \pi}=\frac{\pi}{2}\right. \\
a_{n}=
\end{gathered}
$$

$$
\begin{aligned}
& =\frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \cos (n x) d x+\int_{0}^{\pi} x \cos (n x) d x\right] \\
& =\frac{1}{\pi} \int_{0}^{\pi} x \cos (n x) d x \quad u=x \quad v=\frac{1}{n} \sin (n x) \quad d v=\operatorname{cor}(n x) d x \\
& =\frac{1}{\pi}\left[\left.\frac{x}{n} \sin (n x)\right|_{0} ^{\pi}-\frac{1}{n} \int_{0}^{\pi} \sin (n x) d x\right. \\
& =\frac{1}{\pi}\left[\frac{\pi}{n} \sin (n \pi)-0+\left.\frac{1}{n^{2}} \cos (n x)\right|_{0} ^{\pi}\right. \\
& =\frac{1}{\pi}\left[\frac{1}{n^{2}} \cos (n \pi)-\frac{1}{n^{2}} \cos (0)\right]=\frac{(-1)^{n}-1}{n^{2} \pi}
\end{aligned}
$$

$$
\begin{aligned}
b_{n} & =\frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin (n x) d x \\
& =\frac{1}{\pi}\left[\int_{-\pi}^{0} 0 \sin (n x) d x+\int_{0}^{\pi} x \sin (n x) d x\right] \\
& =\frac{1}{\pi} \int_{0}^{\pi} x \sin (n x) d x \quad u=x \quad d u=d x \\
& =\frac{1}{\pi}\left[\left.\frac{-x}{n} \cos (n x)\right|_{0} ^{\pi}+\frac{1}{n} \int_{0}^{\pi} \cos (n x) d x\right]_{0}^{v=\frac{-1}{n} \cos (n x) d v=\sin (n x) d x} \\
& =\frac{1}{\pi}\left[\frac{-\pi}{n} \cos (n \pi)-0+\left.\frac{1}{n^{2}} \sin (n x)\right|_{0} ^{\pi}\right. \\
& =\frac{-1}{n} \cos (n \pi)+\frac{1}{\pi} \frac{1}{n^{2}} \sin (n \pi)-\frac{1}{\pi} \frac{1}{n^{2}} \operatorname{Sin} /(0)
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{-1}{n}(-1)^{n}=\frac{(-1)^{n+1}}{n} \\
& a_{0}=\frac{\pi}{2}, a_{n}=\frac{(-1)^{n}-1}{n^{2} \pi}, b_{n}=\frac{(-1)^{n+1}}{n} \\
& \text { N.t. } \frac{a_{0}}{2}=\frac{1}{2}\left(\frac{\pi}{2}\right)=\frac{\pi}{4} \\
& f(x)=\frac{\pi}{4}+\sum_{n=1}^{\infty} \frac{(-1)^{n}-1}{n^{2} \pi} \cos (n x)+\frac{(-1)^{n+1}}{n} \sin (n x) \\
&
\end{aligned}
$$

## Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos \left(\frac{n \pi x}{p}\right), \left.\sin \left(\frac{m \pi x}{p}\right) \right\rvert\, n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties
$\int_{-p}^{p} \cos \left(\frac{n \pi x}{p}\right) d x=0$ and $\int_{-p}^{p} \sin \left(\frac{m \pi x}{p}\right) d x=0$ for all $n, m \geq 1$,
$\int_{-p}^{p} \cos \left(\frac{n \pi x}{p}\right) \sin \left(\frac{m \pi x}{p}\right) d x=0$ for all $m, n \geq 1$,
$\int_{-p}^{p} \cos \left(\frac{n \pi x}{p}\right) \cos \left(\frac{m \pi x}{p}\right) d x=\left\{\begin{array}{ll}0, & m \neq n \\ p, & n=m\end{array}\right.$,
$\int_{-p}^{p} \sin \left(\frac{n \pi x}{p}\right) \sin \left(\frac{m \pi x}{p}\right) d x=\left\{\begin{array}{ll}0, & m \neq n \\ p, & n=m\end{array}\right.$.

## Fourier Series on an interval ( $-p, p$ )

The orthogonality relations provide for an expansion of a function $f$ defined on $(-p, p)$ as

$$
f(x)=\frac{a_{0}}{2}+\sum_{n=1}^{\infty}\left(a_{n} \cos \left(\frac{n \pi x}{p}\right)+b_{n} \sin \left(\frac{n \pi x}{p}\right)\right)
$$

where

$$
\begin{aligned}
a_{0} & =\frac{1}{p} \int_{-p}^{p} f(x) d x, \\
a_{n} & =\frac{1}{p} \int_{-p}^{p} f(x) \cos \left(\frac{n \pi x}{p}\right) d x, \quad \text { and } \\
b_{n} & =\frac{1}{p} \int_{-p}^{p} f(x) \sin \left(\frac{n \pi x}{p}\right) d x
\end{aligned}
$$

## Find the Fourier series of $f$

$$
f(x)=\left\{\begin{array}{ll}
1, & -1<x<0 \\
-2, & 0 \leq x<1
\end{array} \quad \text { Here } \quad(-p, p)=(-1,1)\right.
$$



$$
\begin{aligned}
a_{0} & =\frac{1}{1} \int_{-1}^{1} f(x) d x=\int_{-1}^{0} 1 d x+\int_{0}^{1}-2 d x \\
& =\left.x\right|_{-1} ^{0}-\left.2 x\right|_{0} ^{1}=(0-(-1))-2(1-0)=-1 \\
a_{n} & =\frac{1}{1} \int_{-1}^{1} f(x) \cos \left(\frac{n \pi x}{1}\right) d x \\
& =\int_{-1}^{0} \cos (n \pi x) d x+\int_{0}^{1}-2 \cos (n \pi x) d x \\
& =\left.\frac{1}{n \pi} \sin (n \pi x)\right|_{-1} ^{0}-\left.\frac{2}{n \pi} \sin (n \pi x)\right|_{0} ^{1} \\
& =\frac{1}{n \pi} \sin (0)-\frac{1}{n \pi} \sin (-n \pi)-\frac{2}{n \pi} \sin (n \pi)+\frac{2}{n \pi} \sin (0)
\end{aligned}
$$

$$
\begin{aligned}
& =0 \quad a_{n}=0 \text { for } n \geqslant 1 . \\
b_{n} & =\frac{1}{1} \int_{-1}^{1} f(x) \sin \left(\frac{n \pi x}{1}\right) d x \\
& =\int_{-1}^{0} \sin (n \pi x) d x-\int_{0}^{1} 2 \sin (n \pi x) d x \\
& =\left.\frac{-1}{n \pi} \cos (n \pi x)\right|_{-1} ^{0}+\left.\frac{2}{n \pi} \cos (n \pi x)\right|_{0} ^{1} \\
& =\frac{-1}{n \pi}(\cos (0)-\cos (-n \pi))+\frac{2}{n \pi}(\cos (n \pi)-\cos (0)) \\
& =\frac{-1}{n \pi}\left(1-(-1)^{n}\right)+\frac{2}{n \pi}\left((-1)^{n}-1\right)
\end{aligned}
$$

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$$
\begin{aligned}
& =\frac{-1}{n \pi}+\frac{(-1)^{n}}{n \pi}+\frac{2(-1)^{n}}{n \pi}-\frac{2}{n \pi}=\frac{3(-1)^{n}-3}{n \pi} \\
& a_{0}=-1, \quad a_{n}=0, b_{n}=\frac{3\left((-1)^{n}-1\right)}{n \pi} \\
& \frac{a_{0}}{2}=\frac{-1}{2} \text { and } \sin \left(\frac{n \pi x}{p}\right)=\sin (n \pi x)
\end{aligned}
$$

So

$$
f(x)=\frac{-1}{2}+\sum_{n=1}^{\infty} \frac{3\left((-1)^{n}-1\right)}{n \pi} \sin (n \pi x)
$$

## Convergence?

The last example gave the series

$$
f(x)=-\frac{1}{2}+\sum_{n=1}^{\infty} \frac{3\left((-1)^{n}-1\right)}{n \pi} \sin (n \pi x)
$$

This example raises an interesting question: The function $f$ is not continuous on the interval $(-1,1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: what is the connection between $f$ and its Fourier series at the point of discontinuity?

This is the convergence issue mentioned earlier.

## Convergence of the Series

Theorem: If $f$ is continuous at $x_{0}$ in $(-p, p)$, then the series converges to $f\left(x_{0}\right)$ at that point. If $f$ has a jump discontinuity at the point $x_{0}$ in
$(-p, p)$, then the series converges in the mean to the average value

$$
\frac{f\left(x_{0}-\right)+f\left(x_{0}+\right)}{2} \stackrel{\text { def }}{=} \frac{1}{2}\left(\lim _{x \rightarrow x_{0}^{-}} f(x)+\lim _{x \rightarrow x_{0}^{+}} f(x)\right)
$$

at that point.

## Periodic Extension:

The series is also defined for $x$ outside of the original domain $(-p, p)$. The extension to all real numbers is $2 p$-periodic.

Find the Fourier Series for $f(x)=x,-1<x<1$

$$
\begin{aligned}
& a_{0}=\frac{1}{1} \int_{n}^{1} f(x) d x=\int_{-1}^{1} x d x=\left.\frac{x^{2}}{2}\right|_{-1} ^{1} \\
&=\frac{1^{2}}{2}-\frac{(-1)^{2}}{2}=0 \\
& a_{n}=\frac{1}{1} \int_{-1}^{1} f(x) \cos (n \pi x) d x=\int_{-1}^{1} x \cos (n \pi x) d x \\
&=\left.\frac{x}{n \pi} \sin (n \pi x)\right|_{-1} ^{1}+\left.\frac{1}{n^{2} \pi^{2}} \operatorname{cor}(n \pi x)\right|_{-1} ^{1} \\
&=\frac{1}{n \pi} \sin (n \pi)-\frac{1}{n \pi} \sin (-n \pi)+\frac{1}{n^{2} \pi^{2}} \cos (n \pi)-\frac{1}{n^{2} \pi^{2}} \cos (-n \pi)
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{n^{2} \pi^{2}}(-1)^{n}-\frac{1}{n^{2} \pi^{2}}(-1)^{n}=0 \\
b_{n} & =\frac{1}{1} \int_{-1}^{1} f(x) \sin (n \pi x) d x=\int_{-1}^{1} x \sin (n \pi x) d x \\
& =\left.\frac{-x}{n \pi} \cos (n \pi x)\right|_{-1} ^{1}+\left.\frac{1}{n^{2} \pi^{2}} \sin (n \pi x)\right|_{-1} ^{0} \\
& =\frac{-1}{n \pi} \cos (n \pi)-\frac{-(-1)}{n \pi} \cos (-n \pi) \\
& =\frac{-1}{n \pi}(-1)^{n}-\frac{1}{n \pi}(-1)^{n}=\frac{-2}{n \pi}(-1)^{n}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{2}{n \pi}(-1)^{n+1} \\
& a_{0}=0, \quad a_{n}=0, \quad b_{n}=\frac{2(-1)^{n+1}}{n \pi} \\
& f(x)=\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n \pi} \sin (n \pi x)
\end{aligned}
$$

## Symmetry

For $f(x)=x, \quad-1<x<1$

$$
f(x)=\sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n \pi} \sin (n \pi x)
$$

Observation: $f$ is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for $f$.

The following plots show $f, f$ plotted along with some partial sums of the series, and $f$ along with a partial sum of its series extended outside of the original domain $(-1,1)$.


Figure: Plot of $f(x)=x$ for $-1<x<1$


Figure: Plot of $f(x)=x$ for $-1<x<1$ with two terms of the Fourier series.


Figure: Plot of $f(x)=x$ for $-1<x<1$ with 10 terms of the Fourier series


Figure: Plot of $f(x)=x$ for $-1<x<1$ with the Fourier series plotted on $(-3,3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1+1) / 2=0$.

Note $f(x)=x$ is an odd function.
$\operatorname{Cos}(n \pi x)$ is even for every $n$
$\operatorname{Sin}(n \pi x)$ is odd for every $n$


[^0]:    ${ }^{1}$ We'll write $\frac{a_{0}}{2}$ as opposed to $a_{0}$ purely for convenience.

