April 17 Math 2306 sec. 60 Spring 2018

Section 17: Fourier Series: Trigonometric Series

Suppose $f(x)$ is defined for $-\pi < x < \pi$. We would like to know how to write f as a series **in terms of sines and cosines.**

Task: Find coefficients (numbers) $a_0, a_1, a_2, \dots$ and $b_1, b_2, \dots$ such that[1]

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx\right).$$

[1] We'll write $\frac{a_0}{2}$ as opposed to a_0 purely for convenience.

The Fourier Series of $f(x)$ on $(-\pi,\pi)$

The **Fourier series** of the function f defined on $(-\pi,\pi)$ is given by

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos nx+b_n\sin nx\right).$$

Where

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx, \quad \text{and}\\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx
\end{aligned}$$

Some convenient observations

For every integer n

$$\sin(n\pi) = 0$$

For integers n

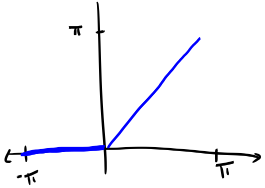
$$\cos(n\pi) = \cos(-n\pi) = (-1)^n$$

$$= \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right|_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2} - 0\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx \qquad u = x \qquad du = dx$$

$$v = \frac{1}{n}\sin(nx) \qquad dv = \cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\cancel{\sin(n\pi)}^{\,0} - 0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left(\frac{1}{n^2}\cos(n\pi) - \frac{1}{n^2}\cos(0)\right) = \frac{1}{\pi n^2}\left((-1)^n - 1\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(nx)\,dx$$

$u = x \qquad du = dx$

$v = \frac{-1}{n}\cos(nx) \qquad dv = \sin(nx)\,dx$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\,dx\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) - 0 + \frac{1}{n^2}\sin(nx)\Big|_0^{\pi}\right.$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n + \frac{1}{n^2}\cancelto{0}{\sin(n\pi)} - \frac{1}{n^2}\cancelto{0}{\sin(0)}\right]$$

$$= \frac{-1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

$$a_0 = \frac{\pi}{2}, \qquad a_n = \frac{(-1)^n - 1}{n^2\pi}, \qquad b_n = \frac{(-1)^{n+1}}{n}$$

$$\frac{a_0}{2} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$$

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right)$$

Fourier Series on an interval $(-p, p)$

The set of functions $\left\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) \middle| n, m \geq 1\right\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\,dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\,dx = 0 \ \text{ for all } \ n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\,dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\,dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\,dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

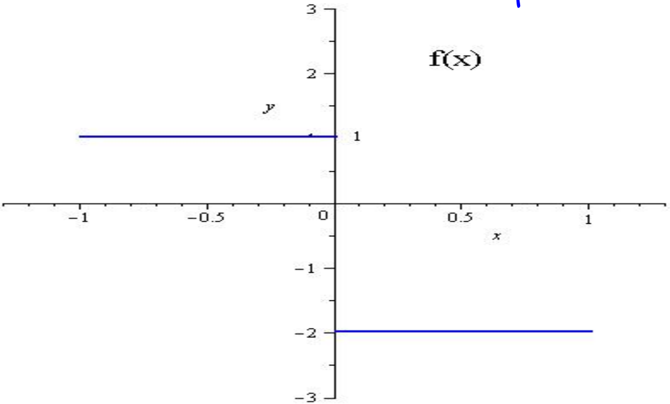
$$\begin{aligned} a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\ a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\ b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx \end{aligned}$$

Find the Fourier series of f

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

$(-p,p) = (-1,1) \Rightarrow p = 1$

So $\frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$



$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} -2\,dx$$

$$= x\Big|_{-1}^{0} - 2x\Big|_{0}^{1} = (0-(-1)) - 2(1-0) = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx = \int_{-1}^{0}\cos(n\pi x)\,dx + \int_{0}^{1} -2\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\underbrace{\sin(0)}_{0} - \frac{1}{n\pi}\underbrace{\sin(-n\pi)}_{0} - \frac{2}{n\pi}\underbrace{\sin(n\pi)}_{0} + \frac{2}{n\pi}\underbrace{\sin(0)}_{0}$$

$a_n = 0$ for all $n \geq 1$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{0} \sin(n\pi x)\,dx + \int_0^1 -2\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$= \frac{-1}{n\pi}\Big(\cos(0) - \cos(-n\pi)\Big) + \frac{2}{n\pi}\Big(\cos(n\pi) - \cos(0)\Big)$$

$$= \frac{-1}{n\pi}\left(1-(-1)^n\right) + \frac{2}{n\pi}\left((-1)^n - 1\right)$$

$$= \frac{-1}{n\pi} + \frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n\pi} - \frac{2}{n\pi}$$

$$= \frac{3(-1)^n - 3}{n\pi} = \frac{3\left((-1)^n - 1\right)}{n\pi}$$

$$a_0 = -1, \quad a_n = 0 \qquad b_n = \frac{3\left((-1)^n - 1\right)}{n\pi}$$

$$\frac{a_0}{2} = \frac{-1}{2}$$

$$\boxed{f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3\left((-1)^n - 1\right)}{n\pi} \sin(n\pi x)}$$

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x\to x_0^-} f(x) + \lim_{x\to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{1}\right)dx$$

$$= \int_{-1}^{1} x\cos(n\pi x)\,dx$$

$$= \frac{x}{n\pi}\sin(n\pi x)\Big|_{-1}^{1} + \frac{1}{n^2\pi^2}\cos(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{1}{n\pi}\sin(n\pi) - \frac{-1}{n\pi}\sin(-n\pi) + \frac{1}{n^2\pi^2}\cos(n\pi) - \frac{1}{n^2\pi^2}\cos(-n\pi)$$

$= 0 \qquad a_n = 0$ for all $n \geq 0$.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x) \sin\left(\frac{n\pi x}{1}\right) dx$$

$$= \int_{-1}^{1} x \sin(n\pi x)\, dx$$

$$= \frac{-x}{n\pi}\cos(n\pi x)\Big|_{-1}^{1} + \frac{1}{n^2\pi^2}\sin(n\pi x)\Big|_{-1}^{1}$$

$$= \frac{-1}{n\pi}\cos(n\pi) - \frac{-(-1)}{n\pi}\cos(-n\pi) + \frac{1}{n^2\pi^2}\sin(n\pi)$$

$$- \frac{1}{n^2\pi^2}\sin(-n\pi)$$

$$= \frac{-1}{n\pi}(-1)^n - \frac{1}{n\pi}(-1)^n = \frac{-2}{n\pi}(-1)^n = \frac{2(-1)^{n+1}}{n\pi}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

$f(x) = x$ is an odd function

$y = 1$ is even

$\cos(n\pi x)$ is even

$\sin(n\pi x)$ is odd

Not surprisingly, the series for odd f only contains odd terms.

Symmetry

For $f(x) = x, \quad -1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f.

The following plots show f, f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

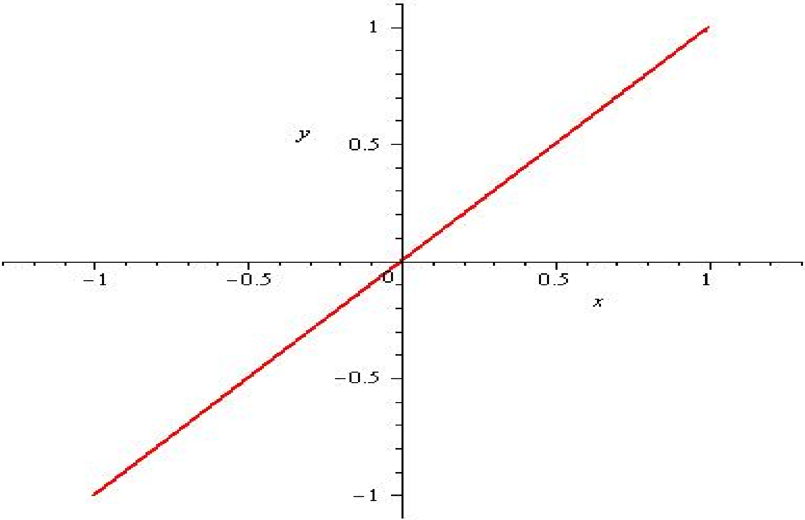


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

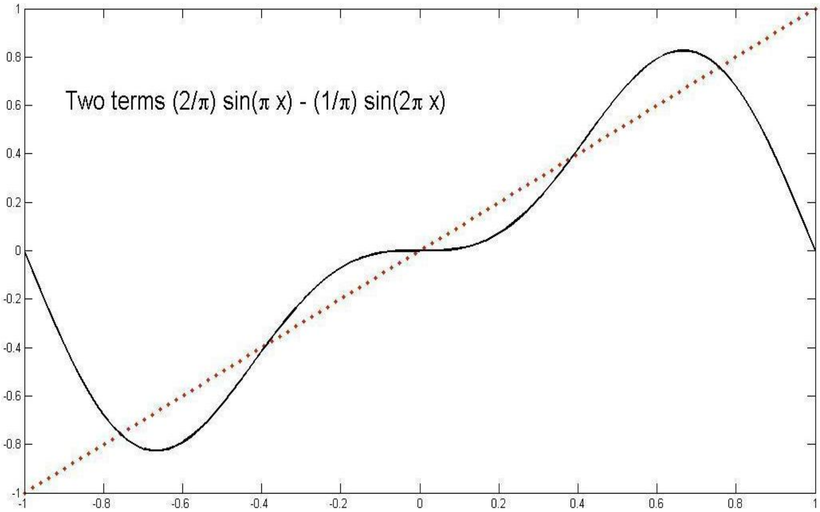


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

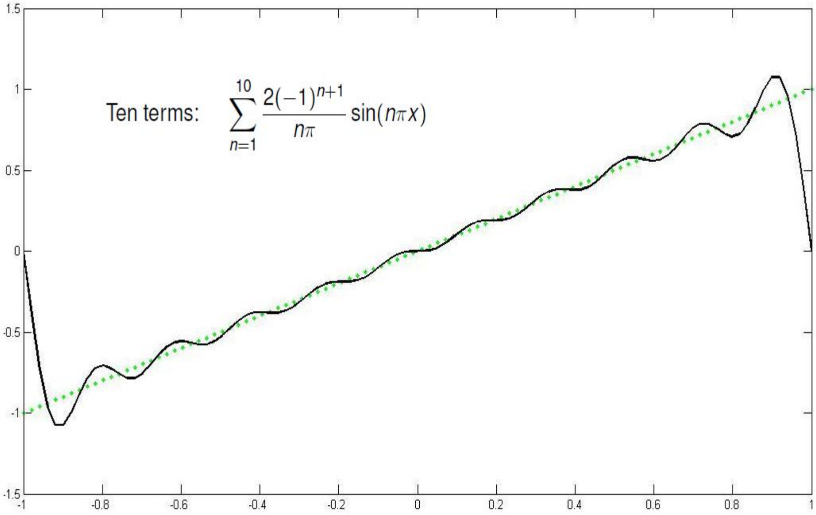


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

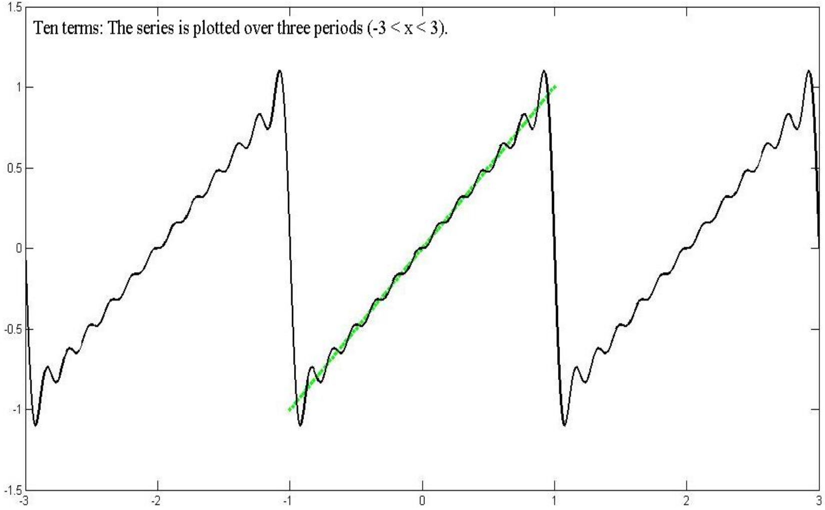


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1+1)/2 = 0$.