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Section 5.1: Eigenvectors and Eigenvalues

Definition: Let A be an $n \times n$ matrix. A nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ is called an **eigenvector** of the matrix A.

A scalar λ such that there exists a nonzero vector $\mathbf{x}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ is called an **eigenvalue** of the matrix A. Such a nonzero vector $\mathbf{x}$ is an *eigenvector corresponding to* λ.

Note that built right into this definition is that the eigenvector **x must be** nonzero!

Eigenspace

Definition: Let A be an $n \times n$ matrix and λ and eigenvalue of A. The set of all eigenvectors corresponding to λ together with the zero vector—i.e. the set

$$\{\mathbf{x} \in \mathbb{R}^n \mid \text{ and } A\mathbf{x} = \lambda\mathbf{x}\},$$

is called the **eigenspace of A corresponding to λ.**

Remark: The eigenspace is the same as the null space of the matrix $A - \lambda I$. It follows that the eigenspace is a subspace of $\mathbb{R}^n$.

Example

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has eigenvalue $\lambda = 2$. Find a basis for the eigenspace of A corresponding to λ.

$A\vec{x} = 2\vec{x}$ $\qquad$ $A\vec{x} = 2I\vec{x}$

$A\vec{x} - 2I\vec{x} = \vec{0} \Rightarrow (A - 2I)\vec{x} = \vec{0}$

$$A - 2I = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix}$$

$\xrightarrow{\text{rref}}$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = \frac{1}{2}x_2 - 3x_3$

x_2, x_3 - free

$$\vec{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

A basis is $\left\{ \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

Matrices with Nice Structure

Theorem: If A is an $n \times n$ triangular matrix, then the eigenvalues of A are its diagonal elements.

Find the eigenvalues of the matrix $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & \pi & 0 \\ -1 & 0 & 1 \end{bmatrix}$

The eigenvalues are $\lambda_1 = 3, \quad \lambda_2 = \pi, \quad \lambda_3 = 1$

Example

Suppose $\lambda = 0$ is an eigenvalue[1] of a matrix A. Argue that A is not invertible.

Since 0 is an eigen value, there is a nonzero eigen vector $\vec{x}$ such that $A\vec{x} = 0\vec{x}$. This $\vec{x}$ is a nontrivial solution to the homogeneous equation

$$A\vec{x} = \vec{0}.$$

By the invertible matrix theorem, A is singular — i.e. not invertible.

[1] Eigenvectors must be nonzero vectors, but it is perfectly legitimate to have a zero eigenvalue!

Theorems

Theorem: A square matrix A is invertible if and only if zero is **not** and eigenvalue.

Theorem: If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are eigenvectors of a matrix A corresponding to distinct eigenvalues, $\lambda_1, \ldots, \lambda_p$, then the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is linearly independent.

Linear Independence

Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of a matrix A with corresponding eigenvalues λ_1 and λ_2 where $\lambda_1 \neq \lambda_2$, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$ for some c_1, c_2.

Let's create two equations from this:

① Mult. by λ_1 : $c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$ $*$

② Mult. by A : $A(c_1\vec{v}_1 + c_2\vec{v}_2) = A\vec{0}$

$$c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = \vec{0}$$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0} \quad **$$

Now, subtract $**$ from $*$

$$c_1\lambda_1\vec{v}_1 + c_2\lambda_1\vec{v}_2 = \vec{0}$$

$$-\left(c_1\lambda_1\vec{v}_1 + c_2\lambda_2\vec{v}_2 = \vec{0} \right)$$

$$c_2\lambda_1\vec{v}_2 - c_2\lambda_2\vec{v}_2 = \vec{0}$$

$$c_2(\lambda_1-\lambda_2)\vec{v}_2 = \vec{0}$$

$\vec{v}_2 \neq \vec{0}$ since its an eigenvector.

$\lambda_1 - \lambda_2 \neq 0$ since $\lambda_1 \neq \lambda_2$

Hence $c_2 = 0$ necessarily.

Going back to the equation $c_1\vec{v}_1 + c_2\vec{v}_2 = \vec{0}$,

since $c_2 = 0$, $\quad c_1\vec{v}_1 = \vec{0}$.

Since $\vec{v}_1 \neq \vec{0}$ (as an eigenvector),

$c_1 = 0$. Hence $\{\vec{v}_1, \vec{v}_2\}$ is linearly

independent.

Section 5.2: The Characteristic Equation

Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$ by appealing to the fact that the equation $A\mathbf{x} = \lambda I_2\mathbf{x}$ can be restated as:

Find a nontrivial solution of the homogeneous equation

$$(A - \lambda I_2)\mathbf{x} = \mathbf{0}.$$

$$A\vec{x} = \lambda\vec{x} \implies A\vec{x} = \lambda I\vec{x} \implies A\vec{x} - \lambda I\vec{x} = \vec{0}$$

$$(A - \lambda I)\vec{x} = \vec{0}$$

Nontrivial $\vec{x}$ requires $A - \lambda I$ is singular.

This is the case if $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2-\lambda & 3 \\ 3 & -6-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (2-\lambda)(-6-\lambda) - 3(3)$$

$$= \lambda^2 + 4\lambda - 21$$

λ has to satisfy $\lambda^2 + 4\lambda - 21 = 0$

$$(\lambda + 7)(\lambda - 3) = 0$$

A has 2 eigenvalues $\lambda_1 = -7$ and $\lambda_2 = 3$.

Theorem (adding more to the invertible matrix theorem)

The $n \times n$ matrix A is invertible if and only if[2]

(s) The number 0 is not an eigenvalue of A.

(t) The determinant of A is nonzero.

If $\lambda = 0$, the equation $\det(A - \lambda I) = 0$

becomes $\det(A) = 0$.

[2] This is nothing new, we're just adding to the list.

Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Example

Find the characteristic equation for the matrix and identify all of its eigenvalues.

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = (5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda)$$

$$= (5-\lambda)^2(3-\lambda)(1-\lambda)$$

This is the characteristic polynomial.

The characteristic equation is

$$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$$

The eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 1$.

Multiplicities

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation. The **geometric multiplicity** is the dimension of its corresponding eigenspace.

Example Find the algebraic and geometric multiplicity of the eigenvalue $\lambda = 5$ of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

.

Algebraic mult. of 5 is 2 since $(5-\lambda)^2$ is a factor.

To find geometric mult. we have to find a basis for the eigen space.

we need the null space of $A-5I$

$$A-5I = \begin{bmatrix} 0 & -2 & 6 & -1 \\ 0 & -2 & -8 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{\text{rref}} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_2 = 0$

$x_3 = 0$

$x_4 = 0$

x_1 - free

Eigen vectors look like

$$\vec{x} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

A basis for the eigen space is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$.

The space has dimension 1.

The geometric multiplicity of $\lambda = 5$ is one.