April 18 Math 1190 sec. 62 Spring 2017

Section 4.1: Related Rates

Motivating Example: A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?

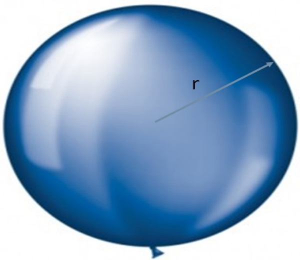


Figure: Spherical Balloon

The $2\,\frac{mm}{sec}$ is the rate of change of r i.e. $\frac{dr}{dt}$

Surface area changes as radius changes

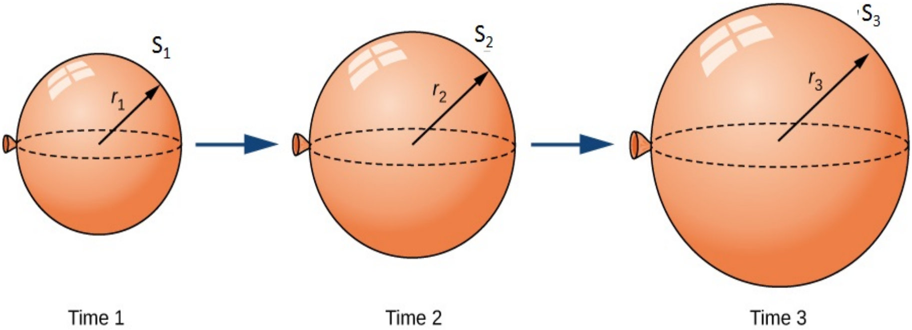


Figure: As a balloon inflates, the radius, surface area, and volume all change with time. Time is independent. Surface area depends on radius which in turn depends on time. $S = S(r) = S(r(t))$

Example Continued...

Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

The chain rule says $\quad \frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$

From $\quad S = 4\pi r^2, \quad \frac{dS}{dr} = 4\pi(2r) = 8\pi r$

So $\quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

This is implicit differentiation, finding $\frac{d}{dt}$ of $S = 4\pi r^2$.

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

We have $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$. We were also given $\frac{dr}{dt} = 2$ mm/sec.

Since $r = 10$ cm at the moment of interest, let's convert that to $r = 100$ mm.

When $r = 100$ mm, $\frac{dS}{dt} = 8\pi(100\text{ mm})\cdot 2\frac{\text{mm}}{\text{sec}} = 1600\pi\frac{\text{mm}^2}{\text{sec}}$.

The surface area is increasing at a rate of

$1600\pi\frac{\text{mm}^2}{\text{sec}}$.

Example

A right circular cone of height h and base radius r has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if r is constant.

Again, the chain rule says

$$\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$$

Since r is assumed constant

$$\left.\frac{dV}{dh} = \frac{\pi}{3} r^2 \cdot 1 = \frac{\pi}{3} r^2 \right\} \Rightarrow \frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$$

Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

Question (b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if h is constant.

$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, $\frac{dV}{dr} = \frac{\pi}{3} h\,(2r)$

(a) $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt}$

(b) $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{\pi}{3} r^2 h \frac{dr}{dt}$

(d) $\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$

And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither r nor h is constant.

r^2h is a product $\Rightarrow$ we need the product rule

$$\frac{dV}{dt} = \frac{dV}{dr}\cdot\frac{dr}{dt} + \frac{dV}{dh}\cdot\frac{dh}{dt}$$

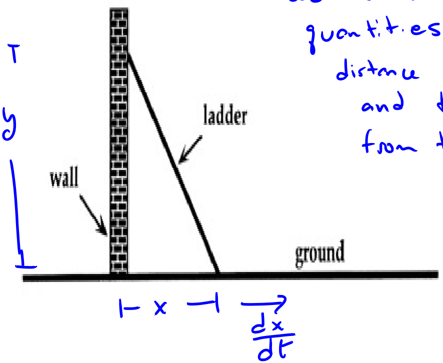
$\frac{dV}{dr} = \frac{2\pi}{3} rh$ and

$\frac{dV}{dh} = \frac{\pi}{3} r^2$ from before

$$\boxed{\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}}$$

Example

A 13 foot ladder rests against a wall. The base of the ladder begins to slide along the ground. At the moment when the base of the ladder is 5 feet from the wall, it is sliding at a rate of 1 inch per second. At what rate is the top of the ladder sliding down the wall at the moment when the base is 5 feet from the wall?

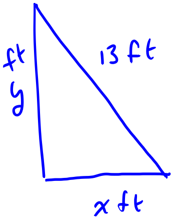


We need variables to represent changing quantities. Let's let x be the distance between the ladder base and the wall, And y be the distance from the ladder top to the ground.

x and y in feet.

given $\frac{dx}{dt} = 1 \frac{\text{in}}{\text{sec}}$

Our question: What is $\frac{dy}{dt}$ when $x = 5$ ft?



We have a right triangle.

$$x^2 + y^2 = 13^2$$

We can "relate the rates":

Implicit diff: $\frac{d}{dt}(x^2+y^2) = \frac{d}{dt}(13^2)$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Let's solve for $\frac{dy}{dt}$: $\quad 2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

$$\frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

We need to know y when $x = 5$ ft.

$$x^2 + y^2 = 13^2 \Rightarrow y^2 = 13^2 - x^2$$

when $x = 5$ f, $y^2 = (13\,ft)^2 - (5\,f)^2 = (169 - 25)\,ft^2$

$$y^2 = 144\ ft^2$$

$y = 12$ ft when $x = 5$ ft

Converting $\frac{dx}{dt}$, $\frac{dx}{dt} = 1\,\frac{in}{sec} \cdot \frac{1\,ft}{12\,in} = \frac{1}{12}\,\frac{ft}{sec}.$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}, \quad \text{when} \quad x = 5\,ft, \quad y = 12\,ft$$

$$\text{and} \quad \frac{dx}{dt} = \frac{1}{12}\frac{ft}{sec}.$$

So when $x = 5\,ft$

$$\frac{dy}{dt} = \frac{-5\,ft}{12\,ft} \cdot \frac{1}{12}\frac{ft}{sec} = \frac{-5}{144}\frac{ft}{sec}.$$

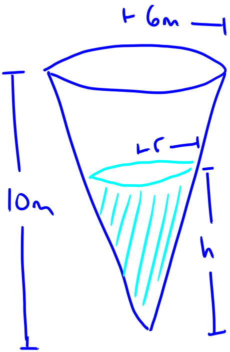
The ladder tip is falling at a rate of $\frac{5}{144}\frac{ft}{sec}$.

General Approach to Solving Related Rates Prob.

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of 50m^3/min. What is the rate at which the height of the water is increasing when the height is 5m?



water in @ 50 m^3/min

At any given time, there is a smaller cone of water inside the reservoir.

Let r and h be the radius and height of the water in meters.

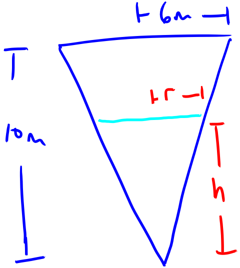
$50 \frac{m^3}{min}$ is a volume rate.

From geometry, $V = \frac{\pi}{3} r^2 h$. Also

$$\frac{dV}{dt} = 50 \ \frac{m^3}{min}$$

Our question is: What is $\frac{dh}{dt}$ when $h = 5m$?

We can use similar triangles to eliminate r.



$$\frac{r}{h} = \frac{6m}{10m} = \frac{3}{5}$$

$$\Rightarrow r = \frac{3}{5} h$$

So $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3} \left(\frac{3}{5} h\right)^2 h = \frac{\pi}{3} \left(\frac{9}{25} h^2\right) h$

$$V = \frac{3\pi}{25} h^3$$

Differentiate:

$$\frac{dV}{dt} = \frac{3\pi}{25} \left(3h^2\right) \cdot \frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{9\pi}{25} h^2 \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{9\pi} \frac{1}{h^2} \frac{dV}{dt}$$

When $h = 5m$

$$\frac{dh}{dt} = \frac{25}{9\pi} \frac{1}{(5m)^2} \cdot 50 \frac{m^3}{min}$$

$$= \frac{25}{9\pi(25m^2)} \, 50 \frac{m^3}{min}$$

$$= \frac{50}{9\pi} \frac{m}{min}$$

The height is increasing at a rate of $\frac{50}{9\pi}$ m/min.