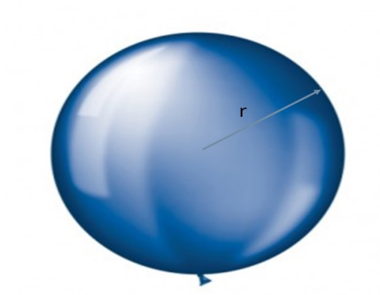
# April 18 Math 1190 sec. 63 Spring 2017

## Section 4.1: Related Rates

**Motivating Example:** A spherical balloon is being filled with air. Suppose that we know that the radius is increasing in time at a constant rate of 2 mm/sec. Can we determine the rate at which the surface area of the balloon is increasing at the moment that the radius is 10 cm?




The $2\frac{mm}{sec}$ is rate of change of radius i.e. $\frac{dr}{dt}$

Figure: Spherical Balloon

# Surface area changes as radius changes

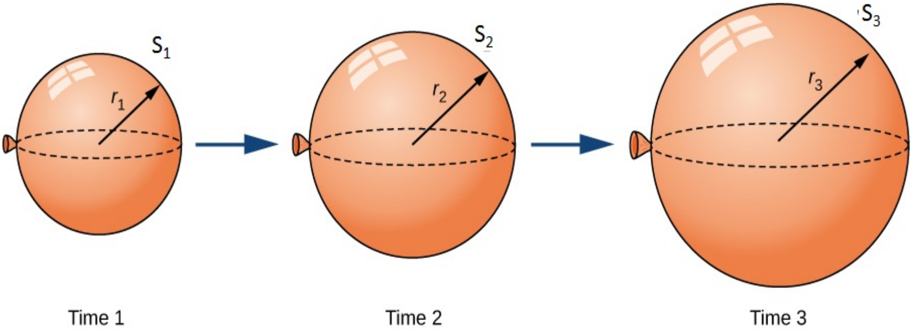



Figure: As a balloon inflates, the radius, surface area, and volume all change with time. Time is independent. Surface area depends on radius which in turn depends on time. $S = S(r) = S(r(t))$

# Example Continued...

Suppose that the radius $r$ and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of time. Write an equation that relates

$$\frac{dS}{dt} \quad \text{to} \quad \frac{dr}{dt}.$$

By the chain rule $\quad \frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}.$

From $S = 4\pi r^2$, $\quad \frac{dS}{dr} = 4\pi(2r) = 8\pi r$

So $\quad \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

* This result is obtained from $S = 4\pi r^2$ using implicit differentiation taking $\frac{d}{dt}$ of both sides. *

Given this result, find the rate at which the surface area is changing when the radius is 10 cm.

We have $\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$ and $\frac{dr}{dt} = 2 \frac{\text{mm}}{\text{sec}}$.

We can convert 10 cm to 100 mm.

When $r = 100$ mm, $\frac{dS}{dt} = 8\pi(100\text{mm}) \cdot 2 \frac{\text{mm}}{\text{sec}}$

$= 1600\pi \frac{\text{mm}^2}{\text{sec}}$

At this moment, the surface area is increasing at a rate of $1600\pi$ $\text{mm}^2$ per second.

# Example

A right circular cone of height $h$ and base radius $r$ has volume

$$V = \frac{\pi}{3} r^2 h.$$

(a) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ if $r$ is constant.

By the chain rule, $\frac{dV}{dt} = \frac{dV}{dh} \cdot \frac{dh}{dt}$

For $r$ constant, $\frac{dV}{dh} = \frac{\pi}{3} r^2 \cdot 1$

So $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dh}{dt}$

# Example Continued...

$$V = \frac{\pi}{3} r^2 h$$

Question (b) Find $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$ if $h$ is constant.

(a) $\frac{dV}{dt} = \frac{2\pi}{3} r \frac{dr}{dt}$

(b) $\frac{dV}{dt} = \frac{\pi}{3} r^2 \frac{dr}{dt}$

(c) $\frac{dV}{dt} = \frac{\pi}{3} r^2 h \frac{dr}{dt}$

(d) $\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt}$

$$\frac{dV}{dt} = \frac{dV}{dr}\,\frac{dr}{dt}$$

and $\frac{dV}{dr} = \frac{\pi}{3} h \left(2r\right)$

# And Continued Further...

$$V = \frac{\pi}{3} r^2 h$$

(c) Find $\frac{dV}{dt}$ in terms of $\frac{dh}{dt}$ and $\frac{dr}{dt}$ assuming neither $r$ nor $h$ is constant.

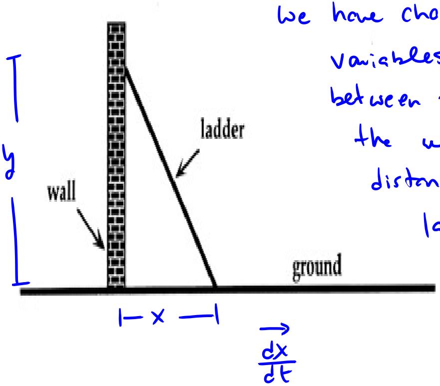
$r^2h$ is a product $\Rightarrow$ we need the product rule.

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} + \frac{dV}{dh}\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{2\pi}{3} rh \frac{dr}{dt} + \frac{\pi}{3} r^2 \frac{dh}{dt}$$

# Example

A 13 foot ladder rests against a wall. The base of the ladder begins to slide along the ground. At the moment when the base of the ladder is 5 feet from the wall, it is sliding at a rate of 1 inch per second. At what rate is the top of the ladder sliding down the wall at the moment when the base is 5 feet from the wall?

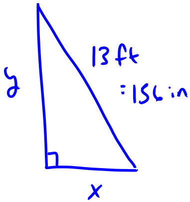



We have changing quantities that need variables. Let $x$ be the distance between the base of the ladder and the wall, and let $y$ be the distance between the top of the ladder and the ground.

Take $x$ and $y$ in inches

Given $\frac{dx}{dt} = 1 \frac{\text{in}}{\text{sec}}$

Our question: What is $\frac{dy}{dt}$ when $x = 60$ inches?




x and y are the legs of a right triangle so

$$x^2 + y^2 = 156^2$$

Use implicit differentiation, find $\frac{d}{dt}$

$$\frac{d}{dt}\left(x^2+y^2\right) = \frac{d}{dt}156^2$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Solving for $\frac{dy}{dt}$

$$2y\frac{dy}{dt} = -2x\frac{dx}{dt} \implies \frac{dy}{dt} = \frac{-2x}{2y}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$$

We need $y$ when $x = 5\,ft = 60\,in$

$$x^2+y^2 = (13\,ft)^2 \implies y^2 = 13^2 ft^2 - 5^2 ft^2 = 144\,ft^2$$

$$y = 12\,ft = 144\,in \quad \text{when } x = 60\,in$$

$$\frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}, \quad \text{given} \quad \frac{dx}{dt} = 1\ \frac{\text{in}}{\text{sec}}$$

when $x = 60$ in, $y = 144$ in and

$$\frac{dy}{dt} = \frac{-60\,\text{in}}{144\,\text{in}} \cdot 1\ \frac{\text{in}}{\text{sec}} = \frac{-5}{12}\ \frac{\text{in}}{\text{sec}}$$

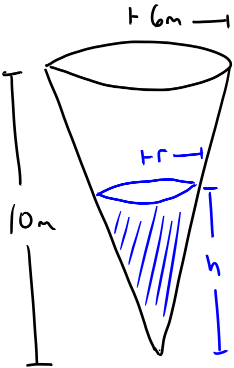
At that moment, the top of the ladder is sliding down at a rate of $\frac{5}{12}$ in per sec.

# General Approach to Solving Related Rates Prob.

- Identifty known and unknown quantities and assign variables.
- Create a diagram if possible.
- Use the diagram, physical science, and mathematics to connect known quantities to those being sought.
- **Relate the rates** of change using implicit differentiation.
- Substitute in known quantities and solve for desired quantities.

# Example

A reservoir in the shape of an inverted right circular cone has height 10m and base radius 6m. If water is flowing into the reservoir at a constant rate of $50\text{m}^3/\text{min}$. What is the rate at which the height of the water is increasing when the height is 5m?




water in @ $50 \frac{m^3}{min}$

Let r and h be the base radius and height of the cone of water inside the reservoir; in meters.

$50\ m^3/min$ is a rate of change of Volume.

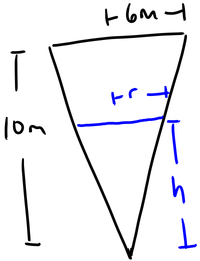
If $V$ is the volume of water, then

$$\frac{dV}{dt} = 50\ \frac{m^3}{min} \quad \text{and} \quad V = \frac{\pi}{3} r^2 h$$

Our question: What is $\frac{dh}{dt}$ when $h = 5m$?

We need to characterize $r$ in terms of $h$.

This is done using similar triangles.




$$\frac{r}{h} = \frac{6m}{10m} = \frac{3}{5} \Rightarrow r = \frac{3}{5} h$$

So $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}\left(\frac{3}{5}h\right)^2 h = \frac{\pi}{3}\,\frac{9}{25}h^2 \cdot h$

$$V = \frac{3\pi}{25} h^3$$

Implicit diff:

$$\frac{dV}{dt} = \frac{3\pi}{25}\left(3h^2\right)\frac{dh}{dt} = \frac{9\pi h^2}{25}\,\frac{dh}{dt}$$

$$\Rightarrow \quad \frac{25}{9\pi h^2}\,\frac{dV}{dt} = \frac{dh}{dt}$$

When $h = 5m$,

$$\frac{dh}{dt} = \frac{25}{9\pi(5m)^2} \cdot 50 \frac{m^3}{min}$$

$$\frac{dh}{dt} = \frac{25}{9\pi(25m^2)} \cdot 50 \frac{m^3}{min}$$

$$= \frac{50}{9\pi} \frac{m}{min}$$

When $h = 5m$, it's increasing at a rate of $\frac{50}{9\pi}$ meters per minute.