April 19 Math 2306 sec. 57 Spring 2018

Section 18: Sine and Cosine Series

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even**.

If $f(-x) = -f(x)$ for all x, then f is said to be **odd**.

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\, dx = 2\int_{0}^{p} f(x)\, dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\, dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x)\, dx$$

and

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right)\, dx.$$

Fourier Series of an Odd Function

If *f* is odd on $(-p, p)$, then the Fourier series of *f* has only sine terms. Moreover

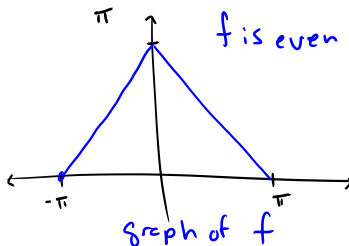$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) \, dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$

π

f is even

$-\pi$ π

graph of f

$b_n = 0$ for all n

$a_0 = \dfrac{2}{\pi} \displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi} \int_0^{\pi} (\pi - x)\,dx$

$\qquad = \dfrac{2}{\pi} \left[\pi x - \dfrac{x^2}{2} \right]_0^{\pi} = \dfrac{2}{\pi} \left[\pi^2 - \dfrac{\pi^2}{2} \right] = \dfrac{2}{\pi} \left(\dfrac{\pi^2}{2} \right) = \pi$

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) \, dx$$

$$= \frac{2}{\pi} \int_0^\pi (\pi - x) \cos(nx) \, dx$$

$$= \frac{2}{\pi} \left[\frac{\pi - x}{n} \sin(nx) \Big|_0^\pi + \frac{1}{n} \int_0^\pi \sin(nx) \, dx \right]$$

$$u = \pi - x \qquad du = -dx$$

$$dv = \cos(nx) \, dx$$

$$v = \frac{1}{n} \sin(nx)$$

$$= \frac{2}{\pi} \left[0 - 0 - \frac{1}{n^2} \cos(nx) \Big|_0^\pi \right]$$

$$= \frac{-2}{\pi n^2} \left(\cos(n\pi) - \cos(0) \right)$$

$$= \frac{-2}{\pi n^2} \left((-1)^n - 1 \right) = \frac{2 \left(1 - (-1)^n \right)}{\pi n^2}$$

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2 \left(1 - (-1)^n \right)}{\pi n^2} \cos(nx)$$

$$a_n = \frac{-2}{n^2\pi}\left((-1)^n - 1\right) = \frac{2}{n^2\pi}\left((-1)^{n+1} + 1\right)$$

We could have written a_n like this $\uparrow$.

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\dfrac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\dfrac{n\pi x}{p}\right)\,dx$.

Half range sine series $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\dfrac{n\pi x}{p}\right)$

where $\quad b_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\left(\dfrac{n\pi x}{p}\right)\,dx$.

Extending a Function to be Odd

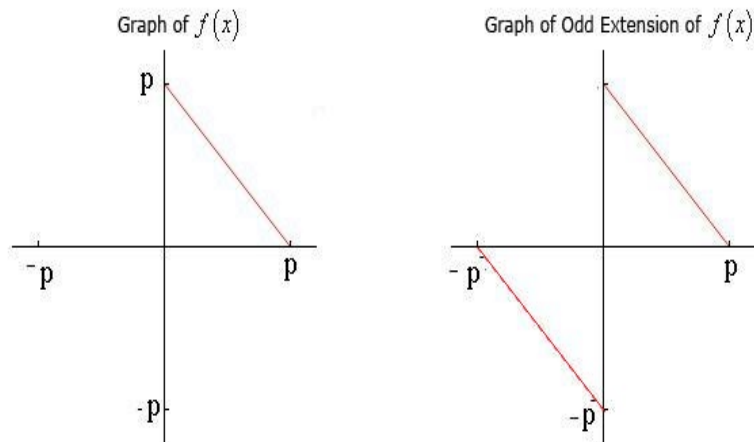
Graph of $f(x)$

Graph of Odd Extension of $f(x)$

Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.
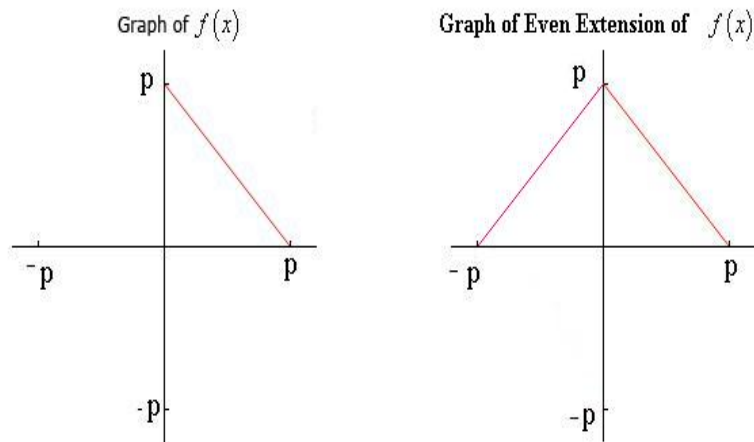
Extending a Function to be Even

Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Find the Half Range Sine Series of *f*

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$P = 2 \qquad \text{so} \qquad \frac{n\pi x}{P} = \frac{n\pi x}{2}$$

$$b_n = \frac{2}{2} \int_0^2 f(x) \, \text{Sin}\left(\frac{n\pi x}{2}\right) dx$$

$$= \int_0^2 (2-x) \, \text{Sin}\left(\frac{n\pi x}{2}\right) dx$$

$$u = 2-x \qquad\qquad du = -dx$$

$$V = \frac{-2}{n\pi} \, \text{Cos}\left(\frac{n\pi x}{2}\right) \qquad dV = \text{Sin}\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{-2(2-x)}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{2}{n\pi} \int_0^2 \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{-2(2-2)}{n\pi} \cos(n\pi) - \frac{-2(2-0)}{n\pi} \cos(0) - \frac{4}{n^2\pi^2} \sin\left(\frac{n\pi y}{2}\right)\Big|_0^2$$

$$= \frac{4}{n\pi}$$

The sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} \sin\left(\frac{n\pi x}{2}\right)$$

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$p = 2$$

$$a_0 = \frac{2}{2} \int_0^2 f(x)\, dx = \int_0^2 (2-x)\, dx = \left(2x - \frac{x^2}{2}\right)_0^2 = 4 - 2 = 2$$

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx = \int_0^2 (2-x) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$u = 2 - x \qquad\qquad du = -dx$$

$$v = \frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right) \qquad dv = \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{2(2-x)}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 + \frac{2}{n\pi} \int_0^2 \sin\left(\frac{n\pi x}{2}\right) dx$$

0

$$= \frac{-4}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= \frac{-4}{n^2\pi^2} \left(\cos(n\pi) - \cos(0) \right)$$

$$= \frac{4}{n^2\pi^2} \left(1 - (-1)^n\right)$$

The Cosine Series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2 \pi^2} \left(1 - (-1)^n\right) \cos\left(\frac{n\pi x}{2}\right)$$
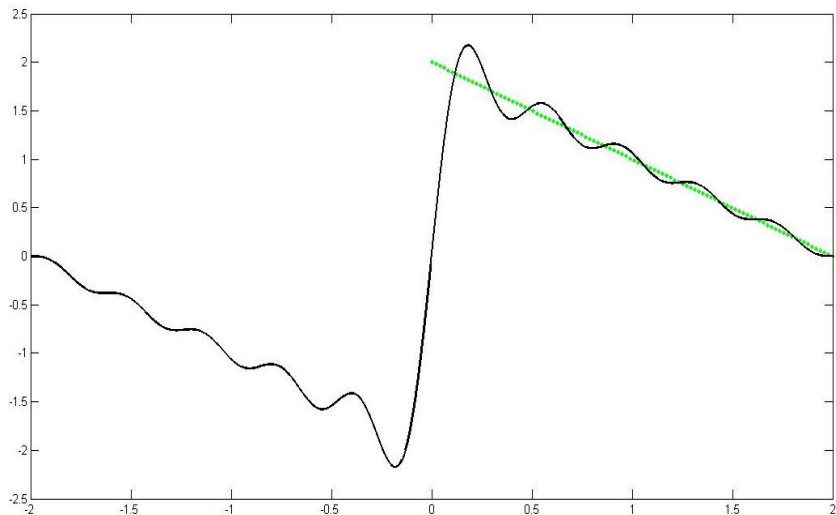
Plots of *f* with Half range series

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.
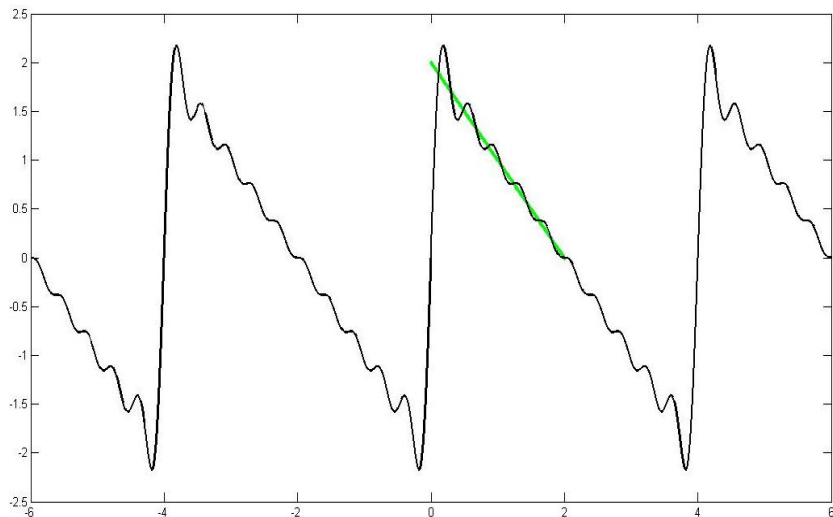
Plots of *f* with Half range series

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$
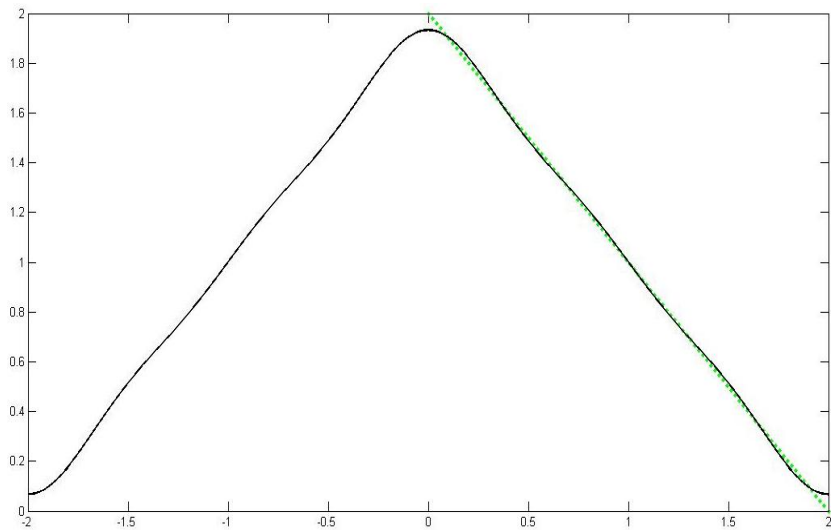
Plots of *f* with Half range series

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.
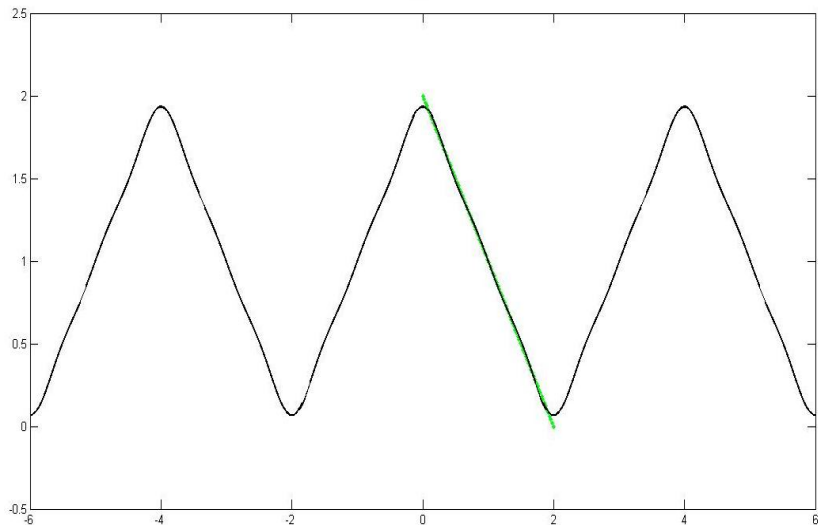
Plots of *f* with Half range series

Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$

Solution of a Differential Equation

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t + 2) = f(t)$ for all $t > 0$. Determine a particular solution x_p for the displacement for $t > 0$.

graph of f
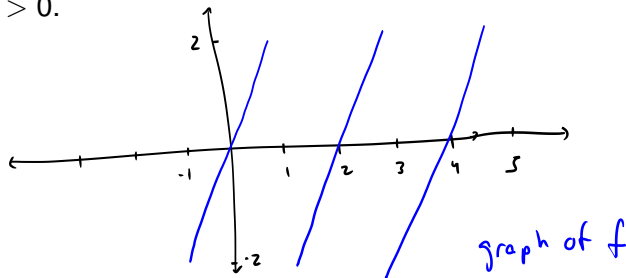
$$mx'' + kx = f(t)$$

$$2x'' + 128x = f(t) \implies x'' + 64x = \frac{1}{2}f(t)$$

We can express f as a sine series.

On Tuesday we found

$$f(t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

We have

$$x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Let's suppose x_p has a sine series

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

Taking derivatives term by term

$$x_p' = \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t)$$

$$x_p'' + 64x_p = \sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\Rightarrow \sum_{n=1}^{\infty} \left(-(n\pi)^2 + 64 \right) B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Equating coefficients of $\sin(n\pi t)$

$$\left(-(n\pi)^2 + 64\right) B_n = \frac{2(-1)^{n+1}}{n\pi}$$

Since $64 - n^2\pi^2 \neq 0$ for all n

$$B_n = \frac{2(-1)^{n+1}}{n\pi\left(64 - n^2\pi^2\right)}$$

So

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64 - n^2\pi^2)} \sin(n\pi t)$$