April 19 Math 2306 sec. 60 Spring 2018

Section 18: Sine and Cosine Series

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x, then f is said to be **even.**

If $f(-x) = -f(x)$ for all x, then f is said to be **odd.**

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 2\int_{0}^{p} f(x)\,dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^{p} f(x)\,dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \quad \times \quad \text{Even} \quad = \quad \text{Even},$$

and

$$\text{Odd} \quad \times \quad \text{Odd} \quad = \quad \text{Even}.$$

While

$$\text{Even} \quad \times \quad \text{Odd} \quad = \quad \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x)\cos(nx)$ is even for all n and $f(x)\sin(nx)$ is odd for all n.

And, if f **is odd** on $(-p, p)$. This tells us that $f(x)\sin(nx)$ is even for all n and $f(x)\cos(nx)$ is odd for all n

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p}\int_0^p f(x)\,dx$$

and

$$a_n = \frac{2}{p}\int_0^p f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p,p)$, then the Fourier series of f has only sine terms. Moreover

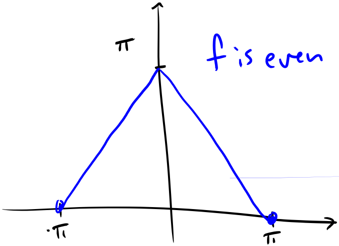
$$f(x)=\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

where

$$b_n=\frac{2}{p}\int_0^p f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ \pi - x, & 0 \le x < \pi \end{cases}$$



f is even

f has no sine terms

$$a_0 = \frac{2}{\pi}\int_0^\pi f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^\pi (\pi - x)\,dx = \frac{2}{\pi}\left[\pi x - \frac{x^2}{2}\right|_0^\pi = \frac{2}{\pi}\left(\pi^2 - \frac{\pi}{2}\right) = \pi$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx)\, dx$$

$$u = \pi - x \qquad du = -dx$$

$$v = \frac{1}{n} \sin(nx) \qquad dv = \cos(nx)\, dx$$

$$= \frac{2}{\pi} \left[\frac{\pi - x}{n} \cancel{\sin(nx)} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} \sin(nx)\, dx \right.$$

0 since $\sin(n\pi) = 0$ for all n

$$= \frac{2}{\pi} \left[\frac{-1}{n^2} \cos(nx) \Big|_0^{\pi} \right. = \frac{-2}{n^2 \pi} \left(\cos(n\pi) - \cos(0) \right)$$

$$a_n = \frac{-2}{n^2\pi}\left((-1)^n - 1\right) = \frac{2}{n^2\pi}\left(1-(-1)^n\right)$$

So

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2\left(1-(-1)^n\right)}{n^2\pi} \cos(nx)$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

Half range cosine series $\quad f(x) = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos\left(\dfrac{n\pi x}{p}\right)$

where $\quad a_0 = \dfrac{2}{p}\displaystyle\int_0^p f(x)\,dx \quad$ and $\quad a_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\cos\left(\dfrac{n\pi x}{p}\right)\,dx.$

Half range sine series $\quad f(x) = \displaystyle\sum_{n=1}^{\infty} b_n \sin\left(\dfrac{n\pi x}{p}\right)$

where $\quad b_n = \dfrac{2}{p}\displaystyle\int_0^p f(x)\sin\left(\dfrac{n\pi x}{p}\right)\,dx.$

Extending a Function to be Odd

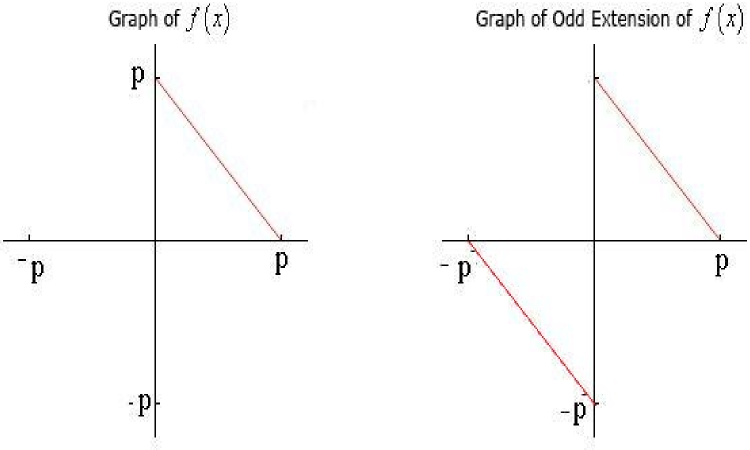


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

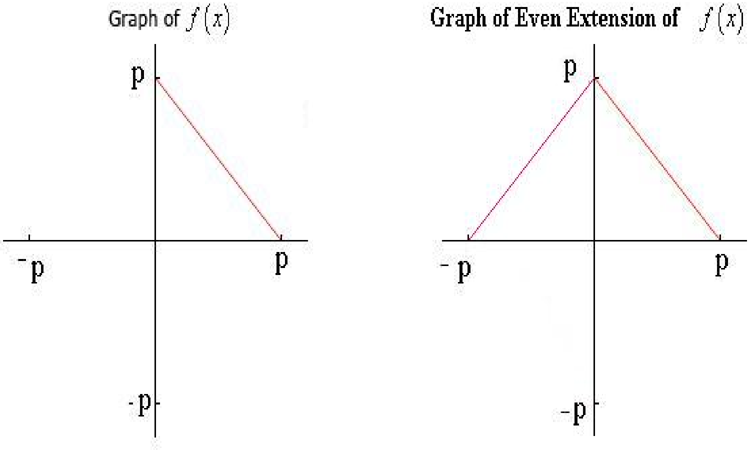


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.

Find the Half Range Sine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$p = 2 \qquad \frac{n\pi x}{p} = \frac{n\pi x}{2}$

$$b_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$u = 2 - x \qquad du = -dx$

$v = \frac{-2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) \qquad dv = \sin\left(\frac{n\pi x}{2}\right)dx$

$$= \frac{-2}{n\pi}(2-x)\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2 - \frac{2}{n\pi}\int_0^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-2}{n\pi}(2-2)\cos(n\pi) - \frac{-2}{n\pi}(2-0)\cos(0) - \frac{4}{n^2\pi^2}\cancelto{0}{\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2}$$

$$= \frac{4}{n\pi}$$

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi}\sin\left(\frac{n\pi x}{2}\right)}$$

Half range sine series

Find the Half Range Cosine Series of f

$$f(x) = 2 - x, \quad 0 < x < 2$$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^2 (2-x)\,dx = 2x - \frac{x^2}{2} = 4 - 2 = 2$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$u = 2 - x \qquad du = -dx$$

$$v = \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) \qquad dv = \cos\left(\frac{n\pi x}{2}\right)dx$$

0

$$= \frac{2}{n\pi}(2-x)\sin\left(\frac{n\pi x}{2}\right)\Big|_0^2 + \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\Big|_0^2$$

$$= \frac{-4}{n^2\pi^2}\left(\cos(n\pi) - \cos(0)\right)$$

$$= \frac{-4}{n^2\pi^2}\left((-1)^n - 1\right) = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

The half range cosine series is

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)\cos\left(\frac{n\pi x}{2}\right)$$

Plots of f with Half range series

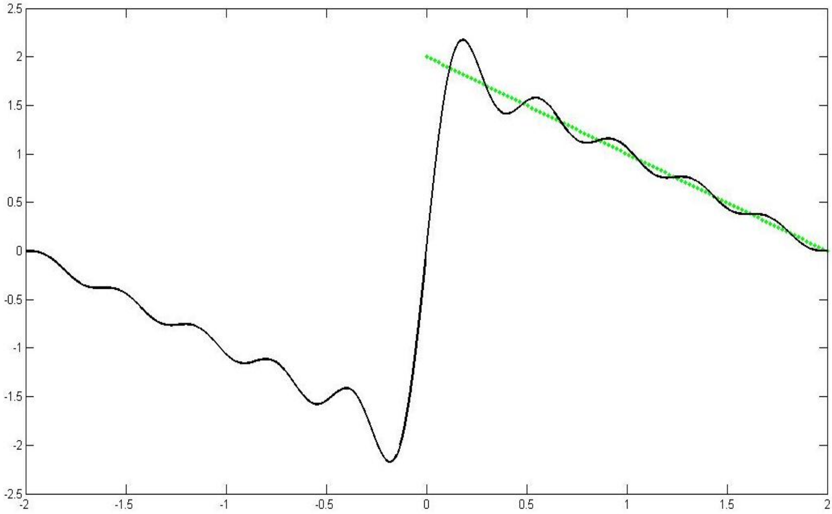


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series.

Plots of f with Half range series

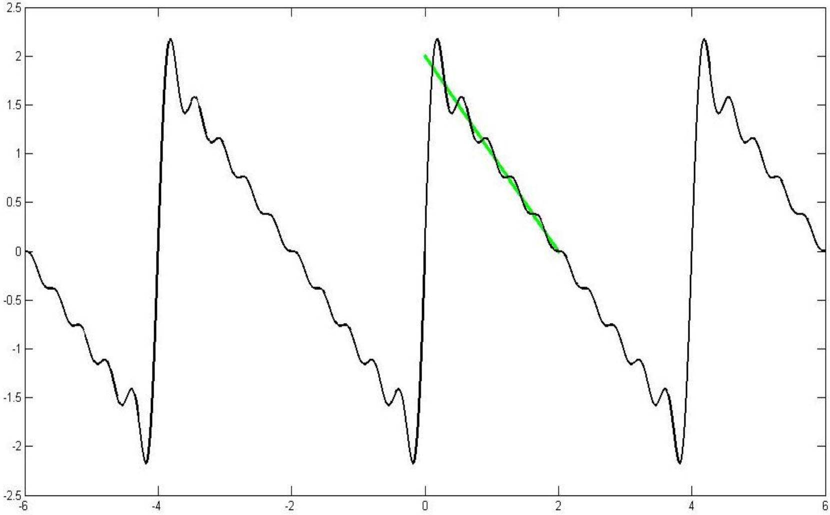


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 10 terms of the sine series, and the series plotted over $(-6, 6)$

Plots of f with Half range series

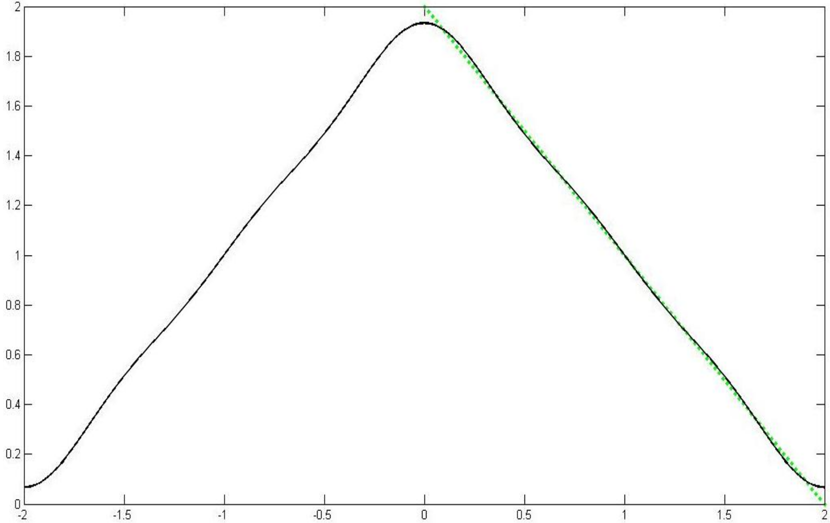


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series.

Plots of f with Half range series

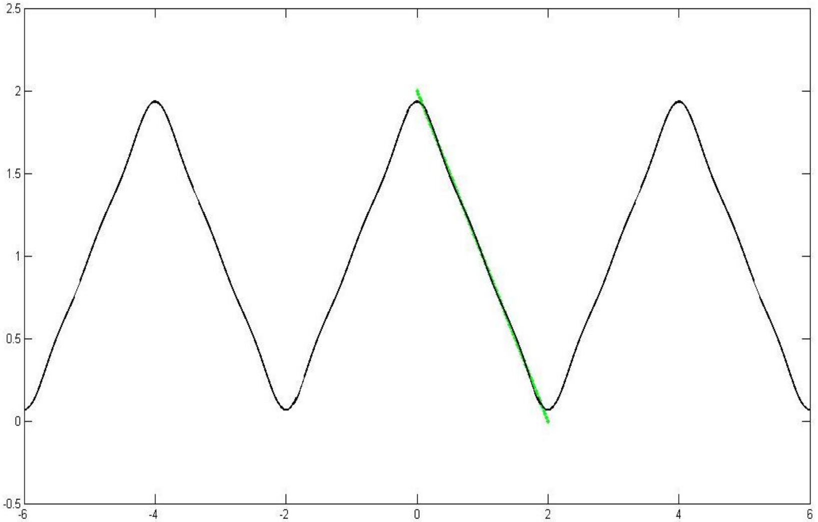


Figure: $f(x) = 2 - x$, $0 < x < 2$ with 5 terms of the cosine series, and the series plotted over $(-6, 6)$