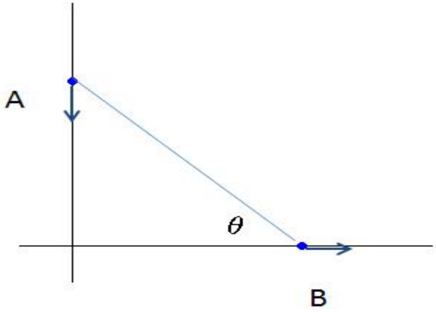
April 20 Math 1190 sec. 63 Spring 2017

Section 4.1: Related Rates

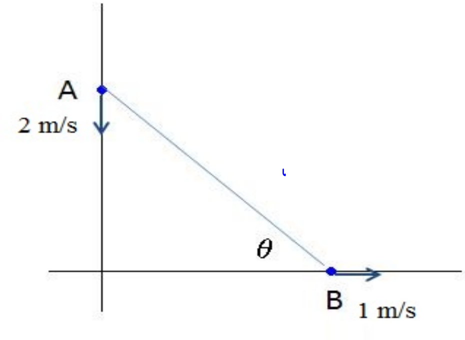
Pedestrians A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2m/sec, and B moves away from the intersection at 1m/sec. Our goal is to determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?



Question

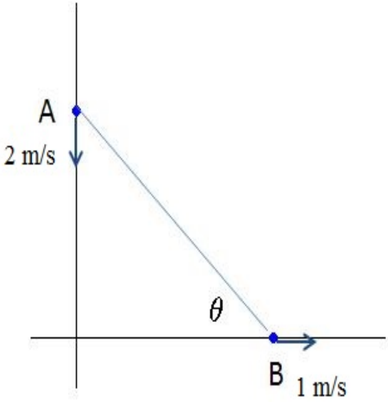
Let $A(t)$ be pedestrian A's position (distance to intersection), and $B(t)$ be pedestrian B's position. Let's make some observations:

(a) **True or False** A is decreasing.
(b) **True or False** B is increasing.



Question

From the diagram, which of the following are the rates of change of A and B (in m/s)?



(a) $\frac{dA}{dt} = -2$ and $\frac{dB}{dt} = 1$

(b) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = -1$

(c) $\frac{dA}{dt} = -2$ and $\frac{dB}{dt} = -1$

(d) $\frac{dA}{dt} = 2$ and $\frac{dB}{dt} = 1$

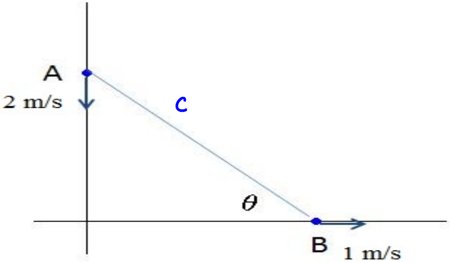
Relating the Rates

The pedestrians' positions and the intersection form a right triangle. So θ, A, and B are related by the equation

$$\tan\theta = \frac{A}{B}$$

Question: Use implicit differentiation to find an expression relating $\frac{d\theta}{dt}$ to the rates of A and B.

$\sin\theta = \frac{A}{c}$, $\cos\theta = \frac{B}{c}$, $\tan\theta = \frac{A}{B}$



Question $\tan\theta = \frac{A}{B}$

The relation between the rates is given by

(a) $\displaystyle \frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$

(b) $\displaystyle \sec^2\left(\frac{d\theta}{dt}\right) = \frac{\frac{dA}{dt}}{\frac{dB}{dt}}$

(c) $\displaystyle \sec^2(\theta)\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$

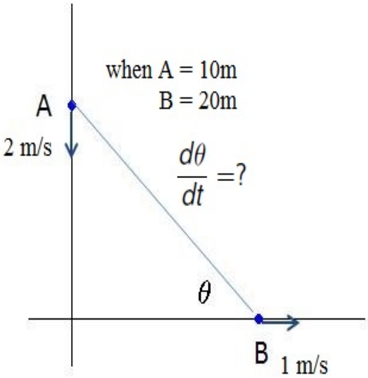
(d) $\displaystyle \sec^2(\theta)\frac{d\theta}{dt} = \frac{A}{B}\frac{dA}{dt} + \frac{A}{B}\frac{dB}{dt}$

$$\frac{d}{dt}\tan\theta = \frac{d}{dt}\left(\frac{A}{B}\right)$$

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$$

The Final Result

Determine the rate at which the angle θ shown in the diagram is changing when A is 10m from the intersection and B is 20 m from the intersection?

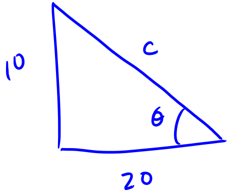


$$\sec^2\theta \frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}$$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2} \cdot \frac{1}{\sec^2\theta}$$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2} \cos^2\theta$$

We need $\cos\theta$ when $A=10$, $B=20$



$$c^2 = 10^2 + 20^2 = 100 + 400 = 500$$

$$\Rightarrow \quad c = \sqrt{500} = 10\sqrt{5}$$

At this moment $\quad \cos\theta = \frac{20}{10\sqrt{5}} = \frac{2}{\sqrt{5}}$

$$\frac{d\theta}{dt} = \frac{\frac{dA}{dt}B - A\frac{dB}{dt}}{B^2}\cos^2\theta$$

$$= \frac{(-2\,\frac{m}{sec})(20m) - (10m)(1\,\frac{m}{sec})}{(20m)^2} \cdot \left(\frac{2}{\sqrt{5}}\right)^2$$

$$\frac{d\theta}{dt} = \frac{(-40-10)\ \frac{m^2}{sec}}{400\ m^2} \cdot \frac{4}{5}$$

$$= \frac{-50}{400} \cdot \frac{4}{5}\ \frac{1}{sec}$$

$$\frac{d\theta}{dt} = \frac{-1}{10}\ \frac{1}{sec}$$

The angle is decreasing at a rate of $\frac{1}{10}$ th radian per second at that time.

Section 4.7: Optimization

Optimization problems arise in every field of study and every industry.

- minimize cost and maximize revenue,
- maximize crop yield,
- minimize driving time,
- maximize volume,
- minimize energy

Often, some constraint (extra condition) must simultaneously be satisfied.

Applied Optimization Example

← we're constrained by this set area condition.

A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of its sides. What dimensions of the outer rectangle will require the smallest total length of fencing and how much fencing will be needed?

we want to minimize the amount of fence

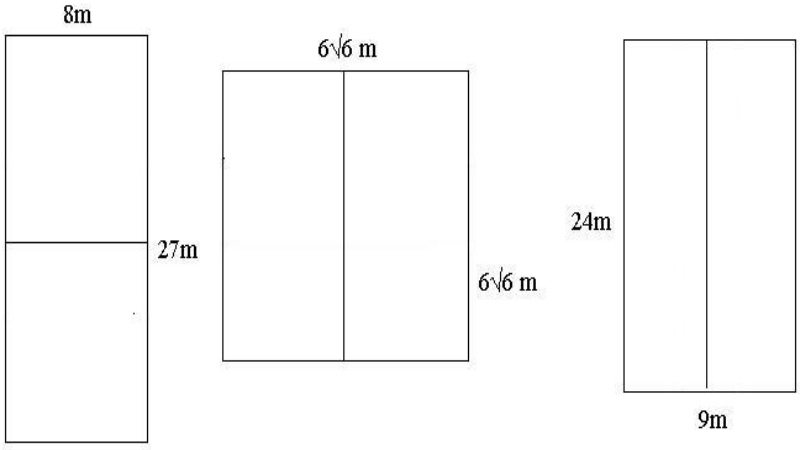


Figure: Different pea patch configuration that all enclose $216m^2$.

Let's start w/ a representative rectangle.

Let l and w be the length and width of our rectangle in meters.

The area $A = lw$, and the amount of fencing is $F = 3w + 2l$.

Our goal is to minimize F. We are constrained by $A = lw = 216$.

We can make F a function of only w by using

$$l = \frac{216}{w}$$

$$F = 3w + 2l = 3w + 2\left(\frac{216}{w}\right) = 3w + \frac{432}{w}$$

$$F(w) = 3w + \frac{432}{w} = 3w + 432w^{-1}.$$

We'll find the critical numbers.

$$F'(w) = 3 - 432w^{-2} = 3 - \frac{432}{w^2}$$

$F'(w)$ is undefined if $w = 0$. But $w > 0$.

$F'(w) = 0 \Rightarrow \quad 0 = 3 - \frac{432}{w^2} \Rightarrow \quad \frac{432}{w^2} = 3$

$$w^2 = \frac{432}{3} = 144 \Rightarrow w = \sqrt{144} = 12 \text{ or}$$
$$w = -\sqrt{144} = -12$$

Since $w > 0$, we have one critical number $w = 12$.

We can use the 2^{nd} derivative test to see if this minimizes F. $\quad F'(w) = 3 - \frac{432}{w^2} = 3 - 432w^{-2}$

$$F''(w) = 0 - 432\left(-2w^{-3}\right) = \frac{864}{w^3}$$

$$F''(12) = \frac{864}{12^3} > 0$$

F would be concave up @ 12

F is minimum when $w = 12$.

When $w = 12\,m$, $l = \frac{216\,m^2}{w\,m} = \frac{216\,m^2}{12\,m} = 18\,m$.

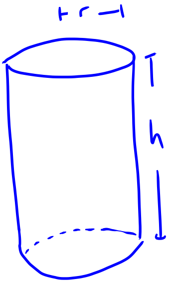
The optimal pea patch is $12m \times 18m$ with the extra piece of fence parallel to the $12m$ sides.

Let's Do One Together

were constrained by this condition

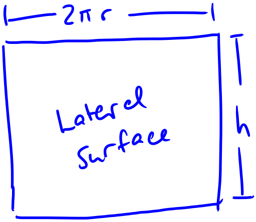
A can in the shape of a right circular cylinder is to have a volume of 128π cubic cm. The material that the top and bottom are made of costs $\$0.20/\text{cm}^2$ and the material that the lateral surface is made of costs $\$0.10/\text{cm}^2$. Find the dimensions of the can that minimize the total cost of production.

minimize a cost



Let r and h be the radius an height of our can in cm.

The top and bottom are disks of area $A_c = \pi r^2$



The area for the lateral surface is

$$A_L = 2\pi r h$$

$$A_c = \pi r^2 \text{ cm}^2 \quad \text{and} \quad A_L = 2\pi r h \text{ cm}^2$$

Question

The total cost $C =$ (cost of lateral surface) + (cost of top & bottom). The cost for the lateral surface was \$0.10/cm^2 while the cost for the top and bottom material is \$0.20/cm^2. The surface area was $S = 2\pi rh + 2\pi r^2$. Which of the following is the cost function?

(a) $C = 2\pi rh + 2\pi r^2$

(b) $C = 0.1(2\pi rh) + 0.2(2\pi r^2)$

(c) $C = 0.2(2\pi rh) + 0.1(2\pi r^2)$

(d) $C = (0.1)(0.2)(2\pi rh + 2\pi r^2)$

Question

The cost appears as a function of two variables, r and h. But we need it to be a function of only one variable.

The volume of the can $V = \pi r^2 h$. We are told it must hold 128π cm^3. Which of the following could be used to express C as a function of r alone?

(a) $h = \dfrac{128}{r}$

(b) $r = \dfrac{128}{\sqrt{h}}$

(c) $h = \dfrac{128}{r^2}$

Question

We can write the cost function in terms of r as

$$C = \frac{25.6\pi}{r} + 0.4\pi r^2$$

Which of the following is the derivative of C with respect to r?

(a) $\dfrac{dC}{dr} = \dfrac{-25.6\pi}{r^2} + 0.8\pi r$

(b) $\dfrac{dC}{dr} = \dfrac{-25.6\pi + 0.8\pi r}{r^2}$

(c) $\dfrac{dC}{dr} = \dfrac{-25.6\pi}{r^2} + 0.4\pi r$

Question

$$\text{Given that} \quad \frac{dC}{dr} = \frac{-25.6\pi}{r^2} + 0.8\pi r,$$

The critical number(s) of C are

$$\frac{25.6\pi}{r^2} = 0.8\pi r$$

(a) 0 and 32

$$\frac{25.6}{0.8} = r^3$$

(b) 0 and $\sqrt[3]{32}$

$$\frac{256}{8} = r^3$$

(c) can't be determined without more information

(d) $\sqrt[3]{32}$

Question

We suspect that the optimal size for the radius, the one that minimizes cost is $\sqrt[3]{32}$. We decide to use the second derivative test to check. We find that

$$\frac{d^2C}{dr^2} = \frac{d}{dr}\left(\frac{-25.6\pi}{r^2} + 0.8\pi r\right) = \frac{51.2\pi}{r^3} + 0.8\pi$$

With no computation, we determine that $r = \sqrt[3]{32}$ is a local minimum because

(a) $C''(r)$ is positive for all positive r, so the graph is concave up.

(b) $C''(r)$ is negative for all positive r, so the graph is concave up.

(c) $C''(r)$ is positive for all positive r, so the graph is concave down.

(d) $C''(r)$ is negative for all positive r, so the graph is concave down.

Question

Since the optimal $r = \sqrt[3]{32}$ and $h = 128/r^2$ our recommendation for minimizing the cost is a can with dimensions

(a) radius of $\sqrt[3]{32}$ cm and height $128/\sqrt[3]{32}$ cm

(b) radius of $\sqrt[3]{32}$ cm and height $128/\sqrt[3]{32^2}$ cm

(c) radius of $\sqrt[3]{32}$ cm and height 4 cm