April 20 Math 2254H sec 015H Spring 2015

Section 11.10: Taylor and Maclaurin Series

Suppose f has a power series representation for $|x-a|<R$. Try to determine a relationship between the coefficients c_n and the values of f and its derivatives as $x=a$.

$$f(x)=c_0+c_1(x-a)+c_2(x-a)^2+c_3(x-a)^3+c_4(x-a)^4+c_5(x-a)^5+\cdots$$

$$f(a)=c_0+c_1\cdot 0+c_2\cdot 0^2+\dots \quad \Rightarrow \quad c_0=f(a)=\frac{f(a)}{0!}$$

$$f'(x)=c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+5c_5(x-a)^4+\dots$$

$$f'(a)=c_1+2c_2\cdot 0+3c_3\cdot 0^2+\dots \quad \Rightarrow \quad c_1=f'(a)=\frac{f'(a)}{1!}$$

$$f''(x) = 2c_2 + 3\cdot 2\, c_3 (x-a) + 4\cdot 3\, a_4 (x-a)^2 + 5\cdot 4\, c_5 (x-a)^3 + \dots$$

$$f''(a) = 2c_2 + 3\cdot 2\cdot c_3 \cdot 0 + \dots$$

$$\Longrightarrow \quad c_2 = \frac{f''(a)}{2} = \frac{f''(a)}{1\cdot 2} = \frac{f''(a)}{2!}$$

$$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4 (x-a) + 5\cdot 4\cdot 3\, c_5 (x-a)^2 + \dots$$

$$f'''(a) = 3\cdot 2\, c_3$$

$$\Longrightarrow \quad c_3 = \frac{f'''(a)}{2\cdot 3} = \frac{f'''(a)}{3!}$$

In general, $c_n = \frac{f^{(n)}(a)}{n!}$

Theorem

Theorem: If f has a power series representation (a.k.a. *expansion*) centered at a,

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad \text{for} \quad |x-a| < R,$$

then the coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

Remark This notation makes use of the traditional convention that the *zeroth* derivative of f is f itself. That is,

$$\frac{f^{(0)}(a)}{0!} = f(a) = c_0.$$

The Taylor Series

Definition: If f has a power series representation centered at a, we can write it as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

This is called the **Taylor series of f centered at** a (or **at** a or **about** a).

Definition: If $a = 0$, the series is called the **Maclaurin series of f**. In this case, the series above appears as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

Example

Determine the Maclaurin series for $f(x) = e^x$. Find its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = e^x \qquad f(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = e^0 = 1$

$f''(x) = e^x \qquad \vdots$

$\vdots$

$f^{(n)}(x) = e^x \qquad f^{(n)}(0) = 1$

So $$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Radius of Convergence

Ratio test: $a_n = \frac{x^n}{n!}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{\cancel{n}+1}}{(n+1)!}\cdot\frac{n!}{\cancel{x^n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{x\,\cancel{n!}}{\cancel{n!}\,(n+1)}\right|$$

$$= \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

$L = 0 < 1$

for all real x

The series converges absolutely for all real x.

The radius of convergence $R = \infty$.

e^x Approximated by terms in its Maclaurin Series

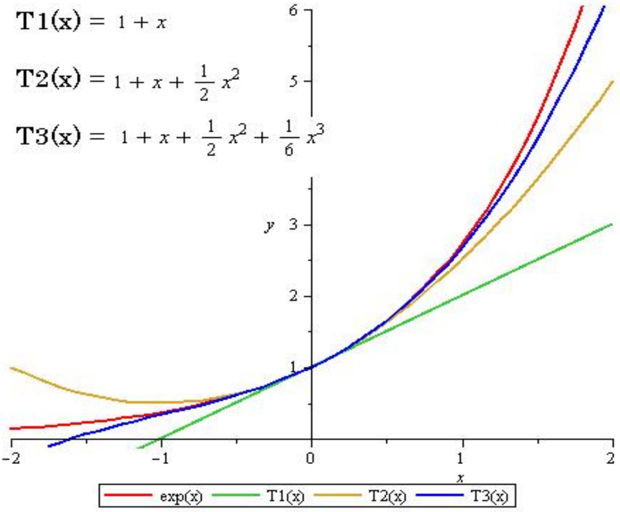

Figure: Plot of f along with the first 2, 3, and 4 terms of the Maclaurin series.

Taylor Polynomials

Definition: Suppose f is at least n times differentiable at $x = a$. The n^{th} **degree Taylor Polynomial of** f **centered at** a, denoted by T_n, is defined by

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

Remark: Note that if f has a Taylor series centered at a, then the Taylor polynomials are what you get if you just take a finite number of terms, and discard the rest.

Remark: A Taylor **series** is like a *polynomial of infinite degree*, but a Taylor **polynomial** will have a well defined finite degree.

Example

Write out the first four Taylor polynomials of $f(x) = e^x$ centered at zero.

Recall $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$T_0(x) = 1$

$T_1(x) = 1 + x$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

$T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

Example

Find the Taylor polynomial of degree $n = 4$ centered at $a = 1$ for $g(x) = e^{3x}$.

$$T_4(x) = \frac{g(1)}{0!} + \frac{g'(1)}{1!}(x-1) + \frac{g''(1)}{2!}(x-1)^2 + \frac{g'''(1)}{3!}(x-1)^3 + \frac{g^{(4)}(1)}{4!}(x-1)^4$$

$g(x) = e^{3x}$

$g'(x) = 3e^{3x}$

$g''(x) = 3^2 e^{3x}$

$g'''(x) = 3^3 e^{3x}$

$g^{(4)}(x) = 3^4 e^{3x}$

$g(1) = e^3$

$g'(1) = 3e^3$

$g''(1) = 9e^3$

$g'''(1) = 27e^3$

$g^{(4)}(1) = 81e^3$

$$T_4(x) = \frac{e^3}{0!} + \frac{3e^3}{1!}(x-1) + \frac{9e^3}{2!}(x-1)^2 + \frac{27e^3}{3!}(x-1)^3 + \frac{81e^3}{4!}(x-1)^4$$

$$T_4(x) = e^3 + 3e^3(x-1) + \frac{9}{2}e^3(x-1)^2 + \frac{9}{2}e^3(x-1)^3 + \frac{27}{8}e^3(x-1)^4$$

Well Known Series and Results

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all real} \quad x$$

A consequence of this is:

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

And with the radius of convergence being infinite, the following limit is useful:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0 \quad \text{for every real number} \quad x$$

Maclaurin Series for $\sin x$

Derive the Maclaurin series of $f(x) = \sin x$. Find its radius of convergence.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$f(x) = \sin x$ $\quad$ $f(0) = 0$

$f'(x) = \cos x$ $\quad$ $f'(0) = 1$

$f''(x) = -\sin x$ $\quad$ $f''(0) = 0$

$f'''(x) = -\cos x$ $\quad$ $f'''(0) = -1$

$f^{(4)}(x) = \sin x$ $\quad$ $f^{(4)}(0) = 0$

$f^{(n)}(0)$ is one of $0, 1$, or -1.

$$\sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 + \frac{-1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 + \frac{-1}{7!}x^7 + \dots$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

Need factor of $(-1)^n$ or $(-1)^{n-1}$

and a formula to pick up odd powers.

If $n = 0, 1, 2, \dots$

taking powers $2n+1$ gives

powers $1, 3, 5, \dots$

$$\boxed{\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}}$$

Radius of Convergence : Ratio test $a_n = \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{x^2 \, (2n+1)!}{(2n+1)! \, (2n+2)(2n+3)} \right|$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+2)(2n+3)} = 0 \qquad L = 0 < 1$$

The radius $R = \infty$; the series converges for all real x.