# April 21 Math 2254H sec 015H Spring 2015

## Section 11.10: Taylor and Maclaurin Series

**Definition:** If $f$ has a power series representation centered at $a$, we can write it as

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \end{aligned}$$

This is called the **Taylor series of $f$ centered at $a$** (or **at $a$** or **about $a$**).

**Definition:** If $a = 0$, the series is called the **Maclaurin series of $f$**. In this case, the series above appears as

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$$

# Taylor Polynomials

**Definition:** Suppose $f$ is at least $n$ times differentiable at $x = a$. The $n^{th}$ **degree Taylor Polynomial of** $f$ **centered at** $a$, denoted by $T_n$, is defined by

$$\begin{aligned} T_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n. \end{aligned}$$

# Maclaurin Series for $\sin x$

Derive the Maclaurin series of $f(x) = \sin x$. Find its radius of convergence.

We found that

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

which is convergent for all real $x$.

# Maclaurin Series for $\cos x$

Use the fact that $\cos x = \frac{d}{dx} \sin x$.

$$\cos x = \frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!} \frac{d}{dx} x^{2n+1} \right]$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1) x^{2n}}{(2n)!\,(2n+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ for all real $x$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

# Well Known Series and Results

$$
\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n, \quad |x| < 1 \\
e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \text{for all x} \\
\sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad \text{for all x} \\
\cos x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \text{for all x} \\
\ln(1+x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}, \quad -1 < x \leq 1 \\
\tan^{-1} x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}, \quad -1 \leq x \leq 1
\end{aligned}
$$

# Compositions, Products and Quotients

If we stay well within the radius of convergence, we can form compositions, products and quotients with Taylor and Maclaurin series.

**Example:** Find a Maclaurin series for $f(x) = e^{-x^2}$.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

for all $t$

Set $t = -x^2$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!}.$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}.$$

# Compositions, Products and Quotients

**Example:**

Find a Maclaurin series representation for the indefinite integral.

$$\int \sin x^2 \, dx$$

$$\sin t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+1}}{(2n+1)!} \quad \text{for all real } t$$

$$\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\int \sin x^2 dx = \int \left( \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!} \right) dx$$

$$= C + \sum_{n=0}^{\infty} \left[ \frac{(-1)^n}{(2n+1)!} \int x^{4n+2}\,dx \right]$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{x^{4n+3}}{4n+3}$$

$$\int \sin x^2\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+3}}{(2n+1)!(4n+3)}$$

# Compositions, Products and Quotients

**Example:**

Use the Maclaurin series for $e^x$ to find a Taylor series for $f(x) = e^x$ centered at $a = -1$.

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} \quad \text{for all } t$$

$$e^x = e^{x+1-1}$$

$$= e^{-1} \cdot e^{x+1}$$

$$= e^{-1} \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(x+1)^n}{en!}$$

# Theorem: The Binomial Series

**Theorem:** For $k$ any real number and $|x| < 1$

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots = \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

Here $\binom{k}{n}$ is read as $k$ *choose* $n$. If is defined by

$$\binom{k}{n} = \frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}.$$

If $k$ is a positive integer, this has the traditional meaning

$$\binom{k}{n} = \frac{k!}{(k-n)!n!}.$$

# Example

Use the Binomial series to find the Taylor polynomial of degree 3 centered at zero for

$$f(x) = \frac{1}{\sqrt[3]{1+x}} = (1+x)^{-\frac{1}{3}} \qquad k = -\frac{1}{3}$$

$$T_3(x) = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3$$

$$k(k-1) = -\frac{1}{3}\left(-\frac{1}{3} - 1\right) = -\frac{1}{3}\left(-\frac{4}{3}\right) = \frac{4}{9}$$

$$k(k-1)(k-2) = \frac{4}{9}\left(-\frac{1}{3} - 2\right) = \frac{4}{9}\cdot\left(-\frac{7}{3}\right) = \frac{-28}{27}$$

$$\frac{4}{9} \cdot \frac{1}{2!} = \frac{2}{9} \, , \qquad \frac{-28}{27} \cdot \frac{1}{3!} = \frac{-28}{27 \cdot 2 \cdot 3} = \frac{-14}{81}$$

$$\boxed{T_3(x) = 1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3}$$

# One More Example

Suppose we have the Taylor series for a function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n(x-3)^n}{2^n(n+2)}$.

Use this to evaluate the following derivative of $f$:

$$f^{(5)}(3)$$

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$c_5 = \frac{f^{(5)}(3)}{5!} \Rightarrow f^{(5)}(3) = 5!\, c_5$$

$$c_5 = \frac{(-1)^5}{2^5(5+2)} = \frac{-1}{32\cdot 7}$$

$$f^{(5)}(3) = 5!\left(\frac{-1}{32\cdot 7}\right) = \frac{-1\cdot 2\cdot 3\cdot 4\cdot 5}{32\cdot 7} = \frac{-15}{56}$$

# Section 11.11: Applications of Taylor Polynomials

**Recall:** The Taylor polynomial of degree $n$ centered at $a$ shares the value of $f$ at $a$ and has the same first $n$ derivative values as $f$ does at the center. Hence $T_n$ approximates the function $f$—typically, the higher the value of $n$, and closer we stay to the center, the better the approximation is.

We can exploit the *nice* nature of polynomials if $f$ itself is somehow difficult to manage!

Wikipedia Page w/ Some Nice Graphics

# Example

Approximate the value of $\sqrt[3]{9}$ by using an appropriate Taylor polynomial of degree 2.

Let $f(x) = \sqrt[3]{x}$. Choose a center "close" to 9 such that $f, f', f''$ are "easy" to evaluate.

Let $a = 8$

$$f(x) \approx T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

for $x \approx a$

$f(x) = x^{1/3}$ $\qquad$ $f(8) = 2$

$f'(x) = \frac{1}{3} x^{-2/3}$ $\qquad$ $f'(8) = \frac{1}{3 \cdot 4} = \frac{1}{12}$

$f''(x) = \frac{-2}{9} x^{-5/3}$ $\qquad$ $f''(8) = \frac{-2}{9 \cdot 32} = \frac{-1}{144}$

$$T_2(x) = 2 + \frac{1}{12}(x-8) - \frac{1}{288}(x-8)^2$$

$$\sqrt[3]{9} \approx T_2(9) = 2 + \frac{1}{12}(9-8) - \frac{1}{288}(9-8)^2$$

$$= \frac{576 + 24 - 1}{288} = \frac{599}{288}$$

# Graph of $f$ and $T_2$ Approximation

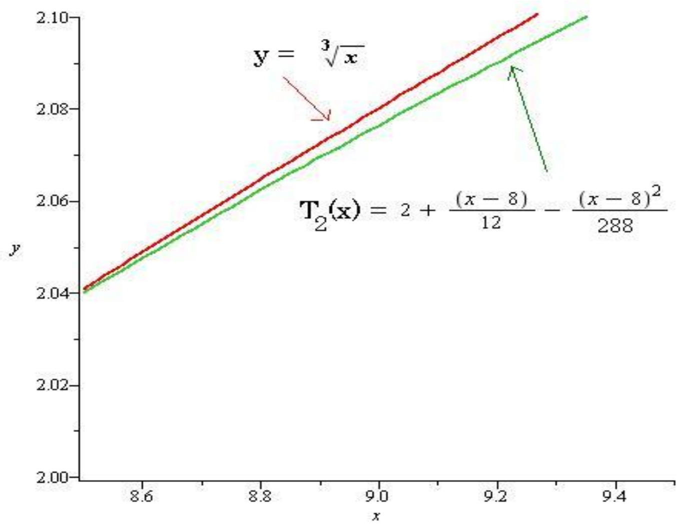


Figure: $f(x) = \sqrt[3]{x}$ together with the second degree Taylor polynomial near the point being approximated.