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Section 5.4: Numerical Differentiation

If we have a scheme to approximate the first derivative $f'(x)$, we're using the notation

$$f'(x) \approx D_h f(x), \quad \text{for step size } h.$$

If we want to approximate $f''(x)$, we'll use a superscript with parentheses

$$f''(x) \approx D_h^{(2)} f(x) \quad \text{for step size } h.$$

For an n^{th} derivative, we write

$$f^{(n)}(x) \approx D_h^{(n)} f(x) \quad \text{for step size } h.$$

Some First Derivative Rules Forward, Backward, and Central Difference

For $h > 0$ we have the names:

$$\text{Forward Difference:} \quad D_h f(x) = \frac{f(x+h) - f(x)}{h}$$

$$\text{Backward Difference:} \quad D_h f(x) = \frac{f(x) - f(x-h)}{h}$$

$$\text{Central Difference:} \quad D_h f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

Errors: Order of a Rule

Defintion: If the error for a particular rule satisfies

$$\text{Err} = Ch^p, \quad \text{for some constants } C \text{ and } p,$$

we will say that the rule is of **order** p.

- The forward and backward difference rules are order $p = 1$.
- The central difference rule is order $p = 2$.

We confirmed both of these using Taylor's theorem.

The Method Undetermined Coefficients

The use of Taylor series expansions can help us to define new numerical differentiation rules as well as analyze the error for a rule.

The Method of Undetermined Coefficients involves setting up a **form** the rule is to take, and then finding out what coefficients are needed.

The Method Undetermined Coefficients an Example

Suppose we wish to approximate a second derivative

$$f''(x) \approx D_h^{(2)} f(x).$$

We begin by deciding how many points to use, such as x, $x+h$, and $x-h$ (or $x+2h$ etc.), then write out a general form.

$$D_h^{(2)} f(x) = Af(x+h) + Bf(x) + Cf(x-h)$$

Then, we determine the values of the unknown coefficients A, B, and C using Taylor series.

A Second Derivative Rule

We sought a rule to approximate $f''(x)$ using three points x, $x+h$ and $x-h$ of the form

$$D^{(2)}f(x) = Af(x+h) + Bf(x) + Cf(x-h).$$

Writing out the Taylor expansions for $Af(x+h)$ and $Cf(x-h)$ we arrived at three equations for our three unknowns

$$\begin{array}{ccccccc} A & + & B & + & C & = & 0 \\ hA & & & - & hC & = & 0 \\ \frac{h^2}{2}A & & & + & \frac{h^2}{2}C & = & 1 \end{array}$$

with solution $A = C = \frac{1}{h^2}$ and $B = \frac{-2}{h^2}$. This gives the rule

$$D_h^{(2)}f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

Determine the Order of The Method Just Found

$$D_h^{(2)} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

From the previous computations, since $A = C = 1/h^2$

$$A\frac{h^3}{6}f'''(x) - C\frac{h^3}{6}f'''(x) = 0.$$

So we can use the Taylor expansions up to degree 3 with error:

$$\frac{1}{h^2}f(x+h) = \frac{1}{h^2}\left(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(c_1)\right)$$

$$\frac{1}{h^2}f(x-h) = \frac{1}{h^2}\left(f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(c_2)\right)$$

Sum

$$\frac{1}{h^2}\left(f(x+h)+f(x-h)\right)=\frac{1}{h^2}\left(2f(x)+h^2f''(x)+\frac{h^4}{24}\left[f^{(4)}(c_1)+f^{(4)}(c_2)\right]\right)$$

$$=\frac{2}{h^2}f(x)+f''(x)+\frac{h^2}{24}\left[f^{(4)}(c_1)+f^{(4)}(c_2)\right]$$

$$\Rightarrow \frac{1}{h^2}\left[f(x+h)+f(x-h)-2f(x)\right]=f''(x)+\frac{h^2}{24}\left[f^{(4)}(c_1)+f^{(4)}(c_2)\right]$$

$$\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x)+(-C)h^2$$

where $-C=\frac{1}{24}\left[f^{(4)}(c_1)+f^{(4)}(c_2)\right]$

$$D_h^{(2)} f(x) = f''(x) - Ch^2$$

$$\Rightarrow \quad f''(x) - D_h^{(2)} f(x) = Ch^2$$

$$\text{Err}\left(D_h^{(2)} f(x)\right) = Ch^2$$

Our rule is an order 2 method.

Section 6.1: Systems of Linear Equations

Definition: A **linear equation** in n variables $x_1, \ldots, x_n$ is one of the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Here, $a_1, \ldots, a_n$ are known constants called the **coefficients**, and b is a known constant.

Examples:

$$2x_1 + 3x_2 - x_3 = 7$$

or

$$x - 2y = 12$$

System of Linear Equations

A **system of linear equations** is two or more linear equations in the same variables—the equations are considered together.

For example:

$$\begin{array}{rcrcr} 3x & + & 2y & = & 9 \\ x & - & 5y & = & -14 \end{array}$$

or

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 3 \\ x_1 & & & + & 2x_3 & = & 3 \end{array}$$

Solution

A system of linear equations may have solutions. A **solution** is an n-tuple of numbers that satisfies all equation in the system simultaneously.

Example: Show that $(x, y) = (1, 3)$ is a solution, and show that $(x, y) = (3, 0)$ is **not** a solution of the system

$$\begin{array}{rcrcr} 3x & + & 2y & = & 9 \\ x & - & 5y & = & -14 \end{array}$$

Check $(1,3)$ $\quad 3\cdot 1 + 2\cdot 3 = 3 + 6 = 9$ ✓

$1 - 5\cdot 3 = 1 - 15 = -14$ ✓

Check $(3,0)$ $\quad 3\cdot 3 - 2\cdot 0 = 9 - 0 = 9$ ✓

$3 - 5\cdot 0 = 3 \neq -14$

Matrices

Definition: A matrix is a rectangular array of numbers. It's **size** (a.k.a. dimension/order) is $m \times n$ (read "*m* by *n*") where m is the number of rows and n is the number of columns the matrix has.

Examples:

$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & 1 & 13 & -4 \\ 12 & -3 & 2 & -2 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 \\ 4 & 4 \\ 3 & -5 \end{bmatrix}$$

3x4 3x2

General System and Matrix Formalism

A system of n equations in n variables has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & & & & & & \vdots & \\ a_{n1}x_1 & + & a_{n2}x_2 & + & \cdots & + & a_{nn}x_n & = & b_n \end{array}$$

We can equate two matrices with this system of equations, a **coefficient matrix** and an **augmented matrix.**

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}.$$

The system is called **homogeneous** if $b_1 = b_2 = \ldots = b_n = 0$.

Example Source of a Linear System

Find a linear system for the following problem and write the coefficient and augmented matrices.

Find a quadratic polynomial $p(x) = ax^2 + bx + c$ through the points $(0, 2)$, $(1, 3)$, and $(2, 10)$.

$p(0) = a \cdot 0^2 + b \cdot 0 + c = 2 \qquad 0a + 0b + c = 2$

$p(1) = a \cdot 1^2 + b \cdot 1 + c = 3 \qquad 1a + 1b + c = 3$

$p(2) = a \cdot 2^2 + b \cdot 2 + c = 10 \qquad 4a + 2b + c = 10$

Coefficient Matrix is

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 1 & 1 & 3 \\ 4 & 2 & 1 & 10 \end{bmatrix}$$

Another Example: Cubic Spline

Suppose we have $h_j = 1$. The equations for the cubic spline numbers M_j were

$$\frac{M_{j-1}}{6} + \frac{2M_j}{3} + \frac{M_{j+1}}{6} = y_{j+1} - 2y_j + y_{j-1}, \quad j = 2, \ldots, n-1.$$

Multiply both sides by 6, and let $b_{j-1} = 6(y_{j+1} - 2y_j + y_{j-1})$.
Then all equations can be written as

$$\begin{array}{ccccccccccc} M_1 & + & 4M_2 & + & M_3 & + & & \cdots & & & = & b_1 \\ & & M_2 & + & 4M_3 & + & M_4 & + & \cdots & & = & b_2 \\ & & & & M_3 & + & 4M_4 & + & M_5 & + \cdots & = & b_3 \\ & & & & & \vdots & & & & & \vdots & \\ & & & & & & M_{n-2} & + & 4M_{n-1} & + M_n & = & b_{n-2} \end{array}$$

Coefficent Matrix for Cubic Spline Equations [1]

$$\begin{bmatrix} 1 & 4 & 1 & 0 & \cdots & & & 0 \\ 0 & 1 & 4 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 & \cdots & 0 \\ & & & \ddots & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \ddots & \\ 0 & \cdots & & & 0 & 1 & 4 & 1 \end{bmatrix}$$

A tridiagonal matrix

[1] This structure is called *tri-diagonal*.

Theorem

Theorem For a linear system of equations, exactly one of the following is true.

- The system has exactly one solution $(x_1, \ldots, x_n)$.
- The system has no solution.
- The system has infinitely many solutions.

A homogeneous system (all right hand sides are zero) always has at least one solution

$$x_1 = x_2 = \ldots = x_n = 0$$

called the **trivial solution**.

Solving a System: Gaussian Elimination

Definition: Two systems are **equivalent** if they have the same solution set.

For example, the following are equivalent:

$$\begin{array}{rcrcr} 3x & + & 2y & = & 9 \\ x & - & 5y & = & -14 \end{array}, \quad \text{and} \quad \begin{array}{rcrcr} 3x & + & 2y & = & 9 \\ & & y & = & 3 \end{array}$$

Note that the solution to the system on the right is fairly obvious.

From $y=3$, we get $3x = 9-2y = 9-2\cdot 3 = 3$

so $x=1$

Solving a System: Gaussian Elimination

We can try to solve a system by obtaining a convenient form for an equivalent system using an augmented matrix. We are allowed to perform three **row operations.**

The following **Row Operations** result in an equivalent system:

i Swap any two rows. Denoted $R_i \leftrightarrow R_j$

ii Multiply a row by any nonzero number $kR_i \rightarrow R_i$

iii Add a nonzero multiple of one row to another row and replace one of these rows with the result. $R_j - kR_i \rightarrow R_j$

Example (a): Gaussian Elimination w/ Back Substitution

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ -x_1 & - & 3x_2 & & & = & 2 \end{array}$$

Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{bmatrix}$$

$R_2 - 2R_1 \to R_2$

$R_3 - (-1)R_1 \to R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{bmatrix}$$

Scratch

	2	2	3	3
	-2	-4	-2	0
	-1	-3	0	2
	1	2	1	0

$R_3 - (\frac{1}{2})R_2 \to R_3$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

$0 \quad -1 \quad 1 \quad 2$

$0 \quad 1 \quad -1/2 \quad -3/2$

The system for this Augmented matrix is equivalent. It is

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ -2x_2 + x_3 &= 3 \\ \tfrac{1}{2}x_3 &= \tfrac{1}{2} \end{aligned}$$

$x_3 = 1$

$$-2x_2 = 3 - x_3$$

$$x_2 = \frac{x_3 - 3}{2} = \frac{1-3}{2} = -1$$

$$x_1 = -2x_2 - x_3 = -2(-1) - 1 = 1$$

$$(x_1, x_2, x_3) = (1, -1, 1)$$

Triangular Matrix

The matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ is called **upper triangular.** A matrix with only zero entries *above* the main diagonal is called **lower triangular.**

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \qquad \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

upper triangular   Lower triangular

Section 6.3: Gaussian Elimination: The Process

We begin with the system

$$\begin{array}{ccccccc} a_{11}x_1 & + & a_{12}x_2 & + & a_{13}x_3 & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & a_{23}x_3 & = & b_2 \\ a_{31}x_1 & + & a_{32}x_2 & + & a_{33}x_3 & = & b_3 \end{array}$$

whose augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

Gaussian Elimination: The Process

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix}$$

We introduce the multipliers

$$m_{21} = \frac{a_{21}}{a_{11}} \quad \text{and} \quad m_{31} = \frac{a_{31}}{a_{11}}$$

and perform the row operations $R_2 - m_{21}R_1$ and $R_3 - m_{31}R_1$ to get new rows 2 and 3. ($^{(2)}$ means *second generation*.)

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{bmatrix}$$

Gaussian Elimination: The Process

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & a_{32}^{(2)} & a_{33}^{(2)} & b_3^{(2)} \end{bmatrix}$$

Then we form another multiplier

$$m_{32} = \frac{a_{32}^{(2)}}{a_{22}^{(2)}}$$

and perform $R_3 - m_{32}R_2$ for a new row 3

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{bmatrix}$$

Gaussian Elimination: The Process

This leaves the augmented matrix for the equivalent system

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22}^{(2)} & a_{23}^{(2)} & b_2^{(2)} \\ 0 & 0 & a_{33}^{(3)} & b_3^{(3)} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} & g_1 \\ 0 & u_{22} & u_{23} & g_2 \\ 0 & 0 & u_{33} & g_3 \end{bmatrix}$$

which can be solved using back substitution.

Gaussian Elimination: The Process

Word of Caution: We've assumed that the numbers we divide by are nonzero! Trouble can occur if one of them is zero or even just *close to* zero.

We can side step the possible problem this causes by using **pivoting.**

Error: An Example

Consider solving the following system using a four digit computer.

$$\begin{array}{ccccccc} 6x_1 & + & 2x_2 & + & 6x_3 & = & -2 \\ x_1 & + & \frac{1}{3}x_2 & + & 1x_3 & = & 1 \\ x_1 & + & 2x_2 & - & x_3 & = & 0 \end{array}$$

The exact solution is $x_1 = -1.6$, $x_2 = 1.8$, $x_3 = 2.0$.

The augmented matrix in our computer is

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 1.000 & .3333 & 1.000 & 1.000 \\ 1.000 & 2.000 & -1.000 & 0.000 \end{bmatrix}$$

with $m_{21} = 0.1667$ and $m_{31} = 0.1667$.

Error: An Example

The second generation in our computer is

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0 & -.0001 & .6667 & 1.333 \\ 0 & 1.667 & -1.333 & .3333 \end{bmatrix}.$$

The new multiplier is (much larger than the numbers we're working with)

$$m_{32} = \frac{1.667}{-.0001} = -16670.$$

The numerical solution we end up with is

$$x_1 = 3.444, \quad x_2 = -15.33, \quad x_3 = 3.998$$

which isn't even close!

Pivoting

A solution is to use swapping of rows called **pivoting**. At each step, look at all possible values for the denominator in our multipliers. Choose the largest one.
For example:

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 1.000 & .3333 & 1.000 & 1.000 \\ 1.000 & 2.000 & -1.000 & 0.000 \end{bmatrix}$$

The three possible denominators are 6, 1 and 1. Choose 6.

$$\begin{bmatrix} 6.000 & 2.000 & 2.000 & -2.000 \\ 0 & -.0001 & .6667 & 1.333 \\ 0 & 1.667 & -1.333 & .3333 \end{bmatrix}.$$

The two possible denominators are $-.0001$ and 1.667. Choose 1.667, so swap rows 2 and 3 before proceding.

Operation Count

To solve the linear system of n equations $A\mathbf{x} = \mathbf{b}$ by Gaussian elimination with back substitution, we had two general processes:

$$A \longrightarrow U, \quad \text{and} \quad b \longrightarrow g \longrightarrow x$$

We can count the number of multiplications, additions, subtractions, divisions involved (# of operations):

$$A \longrightarrow U : \quad \frac{4n^3 + 9n^2 - 7n}{6} \quad \text{operations}$$

$$b \longrightarrow g \longrightarrow x : \quad 2n^2 \quad \text{operations}$$

Operation Count

For example: If A is 5×5, it takes

$$115 + 50 = 165 \quad \text{operations}$$

If A is 10×10, it takes

$$805 + 200 = 1005 \quad \text{operations}$$

Suppose we wish to solve the system $A\mathbf{x} = \mathbf{b}_k$ for $k = 1, 2, \ldots, N$. If

$$A \quad \text{is} \quad 10 \times 10, \quad \text{and} \quad N = 25$$

$$\text{total \# of operations} \quad = \quad 25,125$$

Operation Count

If we can do $A \longrightarrow U$ only once, and $b \longrightarrow g \longrightarrow x$ 25 times, then the total number of operations drops to

$$5805 \quad \text{(about 1/4 as many).}$$

Section 6.4: *LU* Decomposition

Suppose we wish to solve the linear system $A\mathbf{x} = \mathbf{b}$, and we happen to know that

$$A = LU$$

where L is lower triangular, and U is upper triangular.

$$LU\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad L\mathbf{g} = \mathbf{b} \quad \text{and} \quad U\mathbf{x} = \mathbf{g}$$

Example

We wish to solve the system

$$\begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & 3x_3 & = & 12 \\ 4x_1 & - & x_2 & + & 6x_3 & = & 23 \\ -2x_1 & + & 2x_2 & - & 5x_3 & = & -19 \end{array} \qquad A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 6 \\ -2 & 2 & -5 \end{bmatrix}$$

And we know that $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Solve $L\mathbf{g} = \mathbf{b}$ and then $U\mathbf{x} = \mathbf{g}$ where $\mathbf{b} = \begin{bmatrix} 12 \\ 23 \\ -19 \end{bmatrix}$.

$L\mathbf{g} = \mathbf{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 23 \\ -19 \end{bmatrix}$$

$g_1 = 12$ $\quad 2g_1 + g_2 = 23$ $\quad g_2 = 23 - 2g_1 = 23 - 24 = -1$

$-g_1 + g_2 + g_3 = -19$

$g_3 = -19 + g_1 - g_2 = -19 + 12 - (-1) = -6$

$$\therefore \vec{g} = \begin{bmatrix} 12 \\ -1 \\ -6 \end{bmatrix}$$

$U\mathbf{x} = \mathbf{g}$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \\ -6 \end{bmatrix}$$

$$-2x_3 = -6 \quad \Rightarrow \quad x_3 = 3$$

$$x_2 + 0x_3 = -1 \quad \Rightarrow \quad x_2 = -1$$

$$2x_1 - x_2 + 3x_3 = 12$$

$$x_1 = \frac{1}{2}(12 + x_2 - 3x_3) = \frac{1}{2}(12 - 1 - 9) = 1$$

So

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

The LU Factorization

Let A be an $n \times n$ matrix, and suppose that we can do Gaussian elimination with A **without any pivoting.**

That is, we are able to form the necessary multipliers m_{ij} without swapping any rows.

Then we can write A as the product $A = LU$ where

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1n} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n\,n-1} & 1 \end{bmatrix}$$