April 21 Math 2254H sec 015H Spring 2015

Section 11.11: Applications of Taylor Polynomials

Observation: If $T_n(x)$ is the Taylor polynomial of degree n centered at a for the function $f(x)$, then

$$T_n^{(k)}(a) = f^{(k)}(a), \quad \text{for each} \quad k = 0, 1, \dots, n.$$

That is, f and T_n have the same value and the same first n derivatives at the center a.

Example

Find the Taylor polynomial of degree 2 centered at $\frac{\pi}{2}$ for $f(x) = \sin x$. Use this to find an approximation to $\sin 80^\circ$

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$f(x) = \sin x \qquad f(\pi/2) = 1$

$f'(x) = \cos x \qquad f'(\pi/2) = 0$

$f''(x) = -\sin x \qquad f''(\pi/2) = -1$

$$T_2(x) = 1 - \frac{1}{2}(x - \pi/2)^2$$

For $x \approx \frac{\pi}{2}$, $T_2(x) \approx f(x)$

$$80^\circ = \frac{80}{180}\pi = \frac{4\pi}{9} \qquad f\left(\frac{4\pi}{9}\right) \approx T_2\left(\frac{4\pi}{9}\right)$$

$$\sin 80^\circ \approx T_2\left(\frac{4\pi}{9}\right) = 1 - \frac{1}{2}\left(\frac{4\pi}{9} - \frac{\pi}{2}\right)^2$$

$$= 1 - \frac{1}{2}\left(\frac{-\pi}{18}\right)^2$$

$$= 1 - \frac{1}{2}\left(\frac{\pi^2}{324}\right)$$

$$= \frac{648 - \pi^2}{648} \approx 0.985$$

by TI 30 XS

$\sin 80^\circ = 0.985$ to 3 digits

Graph of f and Polynomial Approximations

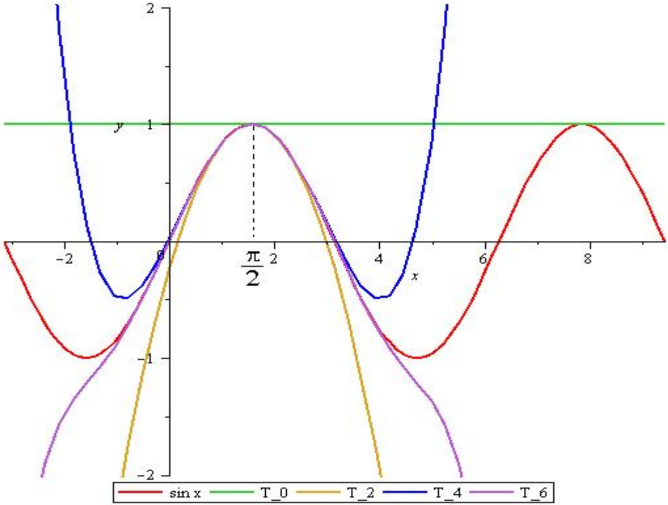


Figure: $f(x) = \sin x$ together with T_n $(n = 0, 2, 4, 6)$ centered at $\frac{\pi}{2}$.

Using Taylor Series to Compute Limits

Use the Maclaurin series for $\sin x$ to verify the well known limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

For $x \approx 0$, $x \neq 0$

$$\frac{\sin x}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!\, x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$$

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) = 1$$

Example

Use an appropriate Taylor series to evaluate the limit

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\cos x - 1 + \frac{x^2}{2} = \left(\cancel{1} - \cancel{\frac{x^2}{2!}} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots\right) - \cancel{1} + \cancel{\frac{x^2}{2}}$$

$$= \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \frac{\frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots}{x^4}$$

$$= \frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \cdots$$

$$\lim_{x\to 0} \frac{\cos x - 1 + \frac{x^2}{2}}{x^4} = \lim_{x\to 0} \left(\frac{1}{4!} - \frac{x^2}{6!} + \frac{x^4}{8!} - \cdots \right)$$

$$= \frac{1}{4!}$$

$$= \frac{1}{24}$$

Error in Taylor Polynomials

Theorem: (Taylor's Inequality)[1] Suppose f is $n+1$ times continuously differentiable on an interval I containing the number a, and let $T_n(x)$ be the Taylor polynomial for f of degree n centered at a. Then for each x in I

$$|f(x) - T_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \quad \text{where} \quad M = \max_{x \text{ in } I} |f^{(n+1)}(x)|.$$

[1] This theorem appears in section 11.10 on page 780 in Stewart.

Example

(i) Find the Taylor polynomial of degree 2 for $f(x) = x \ln x$ centered at $a = 1$.

(ii) Then use Taylor's inequality to estimate the accuracy when T_2 is used to approximate f on the interval $[0.9, 1.1]$.

$$T_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$f(x) = x \ln x$ $\qquad f(1) = 0$

$f'(x) = \ln x + 1$ $\qquad f'(1) = 1$

$f''(x) = \frac{1}{x}$ $\qquad f''(1) = 1$

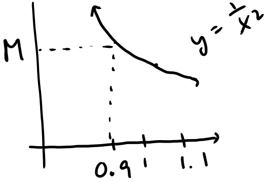
$f'''(x) = \frac{-1}{x^2}$

$$T_2(x) = x - 1 + \frac{1}{2}(x-1)^2$$

ii) $$|f(x) - T_2(x)| \le \frac{M|x-1|^3}{3!} \quad \text{where } M = \max_{x \text{ in } [0.9,\,1.1]} |f'''(x)|$$

Find M: $$|f'''(x)| = \left|\frac{-1}{x^2}\right| = \frac{1}{x^2}$$

$$M = \frac{1}{(0.9)^2} = \left(\frac{10}{9}\right)^2 = \frac{100}{81}$$



Worst case error for $|x-1|^3$ is

$$|0.1|^3 = 10^{-3}$$

The error is "no worse" than

$$|f(x) - T_2(x)| \leq \frac{100}{81} \cdot \frac{10^{-3}}{3!} = \frac{1}{10 \cdot 81 \cdot 6}$$

$$= \frac{1}{810 \cdot 6} = \frac{1}{4860}$$