April 24 Math 3260 sec. 55 Spring 2018

Section 5.2: The Characteristic Equation

Definition: For $n \times n$ matrix A, the expression

$$\det(A - \lambda I)$$

is an n^{th} degree polynomial in λ. It is called the **characteristic polynomial** of A.

Definition: The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A.

Theorem: The scalar λ is an eigenvalue of the matrix A if and only if it is a root of the characteristic equation.

Multiplicities

Definition: The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.

Definition: The **geometric multiplicity** of an eigenvalue is the dimension of its corresponding eigenspace.

Similarity

Definition: Two $n \times n$ matrices A and B are said to be **similar** if there exists an invertible matrix P such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[1].

Theorem: If A and B are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[1] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$\det(B - \lambda I) = \det(P^{-1}AP - \lambda I)$ $\quad * \; I = P^{-1} I P$

$= \det(P^{-1}AP - \lambda P^{-1} I P)$

$= \det(P^{-1}(AP - \lambda I P))$

$= \det(P^{-1}(A - \lambda I)P)$ $\quad * \; \det(XY) = \det(X)\det(Y)$

$= \det(P^{-1}) \det(A - \lambda I) \det(P)$

$= \det(P^{-1}) \det(P) \det(A - \lambda I)$

$* \; \det(P^{-1}) = \dfrac{1}{\det(P)}$

$= 1 \det(A - \lambda I)$

$$\Rightarrow \quad \det(B - \lambda I) = \det(A - \lambda I)$$

B has the same characteristic polynomial as A and hence the same eigenvalues.

Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix P for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

$P^{-1}AP$

$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$

$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$

Example Continued...

Show that the columns of P are eigenvectors of A where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{array}{cc} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ \vec{p}_1 \quad \vec{p}_2 \end{array}.$$

Compute

$$A\vec{p}_1 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{p}_1$ is an eigenvector for $\lambda_1 = 3$

$$A\vec{p}_2 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{p}_2$ is an eigenvector with $\lambda_2 = -4$.

Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix}\right) = (4-\lambda)(2-\lambda)+15$$

$$= \lambda^2 - 6\lambda + 23$$

$$0 = \lambda^2 - 6\lambda + 23 = \lambda^2 - 6\lambda + 9 + 14 = (\lambda-3)^2 + 14$$

$$(\lambda-3)^2 = -14 \implies \lambda - 3 = \pm\sqrt{14}\, i$$

$$\lambda = 3 \pm \sqrt{14}\, i$$

Both eigenvalues are complex valued.

Section 5.3: Diagonalization

Determine the eigenvalues of the matrix D^3 where $D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix}$$

$$D^3 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & (-4)^3 \end{bmatrix}$$

The eigenvalues of D^3 are

$$\lambda_1 = 27 \qquad \lambda_2 = -64$$

Diagonal Matrices

Recall: A matrix D is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

Note: If D is diagonal with diagonal entries d_{ii}, then D^k is diagonal with diagonal entries d_{ii}^k for positive integer k. Moreover, the eigenvalues of D are the diagonal entries.

Powers and Similarity

Show that if A and B are similar, with similarity tranformation matrix P, then A^k and B^k are similar with the same matrix P.

Suppose $B = P^{-1}AP$. Note that

$$B^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}A^2P$$

B^2 is similar to A^2 with the same P.

For integer $k \geq 1$

$$B^k = BB^{k-1} = (P^{-1}AP)(P^{-1}A^{k-1}P)$$

if B^{k-1} is similar to A^{k-1}

$$= P^{-1}AA^{k-1}P$$

$$= P^{-1}A^kP$$

Diagonalizability

Defintion: An $n \times n$ matrix A is called **diagonalizable** if it is similar to a diagonal matrix D. That is, provided there exists a nonsingular matrix P such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

Theorem: The $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors. In this case, the matrix P is the matrix whose columns are the n linearly independent eigenvectors of A.

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\big((-5-\lambda)(1-\lambda)+9\big) - 3\big(-3(1-\lambda)+9\big) + 3\big(-9-3(-5-\lambda)\big)$$

$$= (1-\lambda)(\lambda^2+4\lambda+4) - 3(3\lambda+6) + 3(3\lambda+6)$$

$$= (1-\lambda)(\lambda+2)^2 \quad = -\lambda^3 - 3\lambda^2 + 4$$

The eigenvalues are $\lambda_1 = 1$ (alg. mult. 1)

$\lambda_2 = -2$ (alg. mult. 2)

Find eigenvectors:

$\lambda_1 = 1$ $\quad A - 1I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = x_3$

$x_2 = -x_3$

x_3 - free

A eigenvector is

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -2 \qquad A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_2 - x_3$

x_2, x_3 - free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We can use eigenvectors

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We have 3 lin. ind eigen vectors

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

A P matrix is

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Turns out

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Example

Diagonalize the matrix A if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right)$$

$$\vdots$$

$$= -\lambda^3 - 3\lambda^2 + 4 = (1-\lambda)(\lambda+2)^2$$

$$\lambda_1 = 1, \quad \lambda_2 = -2$$

Find eigen vectors:

$$A - I = \begin{bmatrix} 1 & 4 & -3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$A + 2I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2$

$x_3 = 0$, x_2 - free

An eigenvector is $\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

A is not diagonalizable. It doesn't have enough lin. independent eigenvectors.

Theorem (a second on diagonalizability)

Recall: (sec. 5.1) If λ_1 and λ_2 are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

Theorem: If the $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note: This is a *sufficiency* condition, not a *necessity* condition. We've already seen a matrix with a repeated eigenvalue that was diagonalizable.

Theorem (a third on diagonalizability)

Theorem: Let A be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of λ_k is less than or equal to the algebraic multiplicity of λ_k.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is n—i.e. the sum of dimensions of all eigenspaces is n so that there are n linearly independent eigenvectors.

(c) If A is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for λ_k, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

Remark: The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix.)