# April 24 Math 3260 sec. 56 Spring 2018

## Section 5.2: The Characteristic Equation

**Definition:** For $n \times n$ matrix $A$, the expression

$$\det(A - \lambda I)$$

is an $n^{th}$ degree polynomial in $\lambda$. It is called the **characteristic polynomial** of $A$.

**Definition:**The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of $A$.

**Theorem:** The scalar $\lambda$ is an eigenvalue of the matrix $A$ if and only if it is a root of the characteristic equation.

# Multiplicities

**Definition:** The **algebraic multiplicity** of an eigenvalue is its multiplicity as a root of the characteristic equation.

**Definition:** The **geometric multiplicity** of an eigenvalue is the dimension of its corresponding eigenspace.

# Similarity

**Definition:** Two $n \times n$ matrices $A$ and $B$ are said to be **similar** if there exists an invertible matrix $P$ such that

$$B = P^{-1}AP.$$

The mapping $A \mapsto P^{-1}AP$ is called a **similarity transformation**[1].

**Theorem:** If $A$ and $B$ are similar matrices, then they have the same characteristic equation, and hence the same eigenvalues.

[1] **Note that similarity is NOT related to being row equivalent.**

If $B = P^{-1}AP$, then $\det(B - \lambda I) = \det(A - \lambda I)$

$$\det(B-\lambda I) = \det(P^{-1}AP - \lambda I)$$

$* \quad I = P^{-1}IP$

$$= \det(P^{-1}AP - \lambda P^{-1}IP)$$

$$= \det(P^{-1}(AP - \lambda IP))$$

$$= \det(P^{-1}(A-\lambda I)P)$$

$* \quad \det(XY) = \det(X)\det(Y)$

$$= \det(P^{-1})\det(A-\lambda I)\det(P)$$

$$= \det(A-\lambda I)\det(P^{-1})\det(P)$$

$= \det(A - \lambda I) \cdot 1$

$= \det(A - \lambda I)$

$* \ \det(P^{-1}) = \dfrac{1}{\det(P)}$

That is $\det(B - \lambda I) = \det(A - \lambda I)$.

B has the same characteristic polynomial and hence the same eigenvalues as A.

# Example

Show that $A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$ are similar with the matrix $P$ for the similarity transformation given by $P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$.

We'll show $B = P^{-1}AP$

$$P^{-1} = \frac{1}{\det(P)}\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = -1\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 6 & -12 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$$

# Example Continued...

Show that the columns of $P$ are eigenvectors of $A$ where

$$A = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix} \quad \text{and} \quad P = \begin{array}{c} \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix} \\ \vec{p}_1 \quad \vec{p}_2 \end{array}.$$

$$A\vec{p}_1 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$\vec{p}_1$ is an eigenvector w/ $\lambda_1 = 3$

$$A\vec{p}_2 = \begin{bmatrix} -18 & 42 \\ -7 & 17 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$\vec{p}_2$ is an eigenvector w/ $\lambda_2 = -4$.

# Eigenvalues of a real matrix need not be real

Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 3 \\ -5 & 2 \end{bmatrix}$.

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 4-\lambda & 3 \\ -5 & 2-\lambda \end{bmatrix}\right) = (4-\lambda)(2-\lambda)+15$$

$$= \lambda^2 - 6\lambda + 23$$

$$0 = \lambda^2 - 6\lambda + 23 = \lambda^2 - 6\lambda + 9 + 14 = (\lambda-3)^2 + 14$$

$$(\lambda-3)^2 = -14 \Rightarrow \lambda - 3 = \pm\sqrt{14}\, i$$

$$\lambda = 3 \pm \sqrt{14}\, i$$

# Section 5.3: Diagonalization

Determine the eigenvalues of the matrix $D^3$ where $D = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}$.

$$D^2 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix}$$

$$D^3 = DD^2 = \begin{bmatrix} 3 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} 3^2 & 0 \\ 0 & (-4)^2 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & (-4)^3 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = 3^3$ and $\lambda_2 = (-4)^3$

# Diagonal Matrices

**Recall:** A matrix $D$ is diagonal if it is both upper and lower triangular (its only nonzero entries are on the diagonal).

**Note:** If $D$ is diagonal with diagonal entries $d_{ii}$, then $D^k$ is diagonal with diagonal entries $d_{ii}^k$ for positive integer $k$. Moreover, the eigenvalues of $D$ are the diagonal entries.

# Powers and Similarity

Show that if $A$ and $B$ are similar, with similarity tranformation matrix $P$, then $A^k$ and $B^k$ are similar with the same matrix $P$.

Suppose $B = P^{-1}AP$. Then

$$B^2 = (P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}AIAP = P^{-1}AAP$$

$$B^2 = P^{-1}A^2P$$

For $k \geq 1$

$$B^k = BB^{k-1} = P^{-1}AP(P^{-1}A^{k-1}P)$$

if $B^{k-1}$ is similar to $A^{k-1}$

$$= P^{-1}AA^{k-1}P = P^{-1}A^kP$$

# Diagonalizability

**Defintion:** An $n \times n$ matrix $A$ is called **diagonalizable** if it is similar to a diagonal matrix $D$. That is, provided there exists a nonsingular matrix $P$ such that $D = P^{-1}AP$—i.e. $A = PDP^{-1}$.

**Theorem:** The $n \times n$ matrix $A$ is diagonalizable if and only if $A$ has $n$ linearly independent eigenvectors. In this case, the matrix $P$ is the matrix whose columns are the $n$ linearly independent eigenvectors of $A$.

# Example

Diagonalize the matrix $A$ if possible. $A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Find the eigen values

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right)$$

$$= (1-\lambda)\Big((-5-\lambda)(1-\lambda)+9\Big) - 3\Big(-3(1-\lambda)+9\Big)$$

$$+3\Big(-9-3(-5-\lambda)\Big)$$

$= (1-\lambda)\left(\lambda^2+4\lambda+4\right) - 3\left(3\lambda+6\right) + 3\left(3\lambda+6\right)$

$= (1-\lambda)(\lambda+2)^2 \quad = -\lambda^3 - 3\lambda^2 + 4$

we have 2 eigenvalues, $\lambda_1 = 1$, $\lambda_2 = -2$.

alg. multiplicity 1

alg. multiplicity 2

Find eigenvectors:

$\lambda_1 = 1 \qquad A - I = \begin{bmatrix} 0 & 3 & 3 \\ -3 & -6 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

An eigenvector is

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$x_1 = x_3$

$x_2 = -x_3$

$x_3$ - free

$\lambda_2 = -2$

$$A + 2I = \begin{bmatrix} 3 & 3 & 3 \\ -3 & -3 & -3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$x_1 = -x_2 - x_3$

$x_2, x_3$ - free

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Two lin. independent eigenvectors are

$$\vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We have 3 lin. independent eigenvectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We can take $P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then

$$P^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \qquad D = P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

# Example

Diagonalize the matrix $A$ if possible. $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$

Find eigenvalues

$$\det(A-\lambda I) = \det\left(\begin{bmatrix} 2-\lambda & 4 & 3 \\ -4 & -6-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix}\right)$$

$$\vdots$$

$$= -\lambda^3 - 3\lambda^2 + 4$$

$$= (1-\lambda)(\lambda+2)^2$$

The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$

Find eigenvectors:

$\lambda_1 = 1$ $\quad A - I = \begin{bmatrix} 1 & 4 & 3 \\ -4 & -7 & -3 \\ 3 & 3 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

an eigenvector is $\vec{x}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = -2 \qquad A + 2I = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & 3 \\ 3 & 3 & 3 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 = -x_2$

$x_3 = 0$

$x_2$ - free

The eigenvectors are

$$\vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

A doesn't posses three lin. independent eigen vectors.

A is not diagonalizable.

# Theorem (a second on diagonalizability)

**Recall:** (sec. 5.1) If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of a matrix, the corresponding eigenvectors are linearly independent.

**Theorem:** If the $n \times n$ matrix $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

**Note:** This is a *sufficiency* condition, not a *necessity* condition. We've already seen a matrix with a repeated eigenvalue that was diagonalizable.

# Theorem (a third on diagonalizability)

**Theorem:** Let $A$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_p$.

(a) The geometric multiplicity (dimension of the eigenspace) of $\lambda_k$ is less than or equal to the algebraic multiplicity of $\lambda_k$.

(b) The matrix is diagonalizable if and only if the sum of the geometric multiplicities is $n$—i.e. the sum of dimensions of all eigenspaces is $n$ so that there are $n$ linearly independent eigenvectors.

(c) If $A$ is diagonalizable, and $\mathcal{B}_k$ is a basis for the eigenspace for $\lambda_k$, then the collection (union) of bases $\mathcal{B}_1, \ldots, \mathcal{B}_p$ is a basis for $\mathbb{R}^n$.

**Remark:** The union of the bases referred to in part (c) is called an **eigenvector basis** for $\mathbb{R}^n$. (Of course, one would need to reference the specific matrix. )