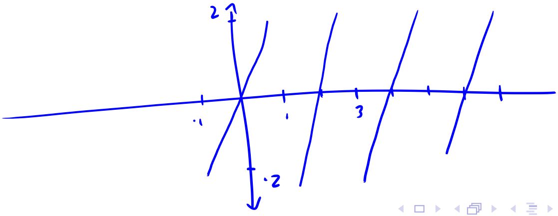
# April 26 Math 2306 sec. 60 Spring 2018

## Section 18: Sine and Cosine Series

An undamped spring mass system has a mass of 2 kg attached to a spring with spring constant 128 N/m. The mass is driven by an external force $f(t) = 2t$ for $-1 < t < 1$ that is 2-periodic so that $f(t+2) = f(t)$ for all $t > 0$. Determine a particular solution $x_p$ for the displacement for $t > 0$.

$mx'' + kx = f(t)$ $\quad m = 2 \quad k = 128$



We can express $f$ as a Fourier Series.

From April $17^{th}$

$$f(t) = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

The ODE is

$$2x'' + 128x = \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\Rightarrow \quad x'' + 64x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

Let's look for $x_p$ in the form

$$x_p = \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

Supposing we can take derivatives term by term

$$x_p' = \sum_{n=1}^{\infty} n\pi B_n \cos(n\pi t)$$

$$x_p'' = \sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t)$$

Substitute

$$\sum_{n=1}^{\infty} -(n\pi)^2 B_n \sin(n\pi t) + 64 \sum_{n=1}^{\infty} B_n \sin(n\pi t)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty} \left( -(n\pi)^2 B_n \sin(n\pi t) + 64 B_n \sin(n\pi t) \right)$$

$$= \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

$$\sum_{n=1}^{\infty}\left(-(n\pi)^2+64\right)B_n \sin(n\pi t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi t)$$

These are equal provided

$$\left(-(n\pi)^2+64\right)B_n = \frac{2(-1)^{n+1}}{n\pi}$$

Since $64-(n\pi)^2 \neq 0$ for all $n$

$$B_n = \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)}$$

So the particular soln.

$$x_p = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi(64-n^2\pi^2)} \sin(n\pi t)$$