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Section 6.4: LU Decomposition

Suppose we wish to solve the linear system $A\mathbf{x} = \mathbf{b}$, and we happen to know that

$$A = LU$$

where L is lower triangular, and U is upper triangular.

$$LU\mathbf{x} = \mathbf{b} \quad \Longleftrightarrow \quad L\mathbf{g} = \mathbf{b} \quad \text{and} \quad U\mathbf{x} = \mathbf{g}$$

The LU Factorization

Let A be an $n \times n$ matrix, and suppose that we can do Gaussian elimination with A **without any pivoting.**

That is, we are able to form the necessary multipliers m_{ij} without swapping any rows.

Then we can write A as the product $A = LU$ where

$$U = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ 0 & u_{22} & \cdots & u_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & u_{1n} \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{21} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ m_{n1} & \cdots & m_{n\,n-1} & 1 \end{bmatrix}$$

Example

Compute L and U for the matrix in the previous example

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 6 \\ -2 & 2 & -5 \end{bmatrix}$$

Use Gaussian Elimination to go from A to $\mathcal{U}$.

$R_2 - 2R_1 \to R_2$ $\quad m_{21} = \frac{4}{2} = 2$

$R_3 - (-1)R_1 \to R_3$ $\quad m_{31} = \frac{-2}{2} = -1$

Set up for matrix L

$$\begin{bmatrix} & & \\ 2 & & \\ -1 & 1 & \end{bmatrix}$$

Scratch

4	-1	6	
-4	2	-6	← -2R1
-2	2	-5	
2	-1	3	← 1R1

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix}$$

$R_3 - 1R_2 \to R_3 \quad m_{32} = \frac{1}{1} = 1$

Scratch

$$\begin{matrix} 0 & 1 & -2 \\ 0 & -1 & 0 \end{matrix} \quad \leftarrow -1R_2$$

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$ This is U

Take the matrix of multipliers, add 1 to the main diagonal, zeros above and that is L

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verify $LU = A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -1 & 6 \\ -2 & 2 & -5 \end{bmatrix}$$

A as expected.

Example

Compute L and U for the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_2 - \frac{1}{2}R_1 \to R_2 \qquad m_{21} = \frac{1}{2} = \frac{1}{2}$

$R_3 - 0R_1 \to R_3 \qquad m_{31} = \frac{0}{2} = 0$

$R_4 - 0R_1 \to R_4 \qquad m_{41} = \frac{0}{2} = 0$

Set up for L

$$\begin{bmatrix} & & & \\ \frac{1}{2} & & & \\ 0 & 2/3 & & \\ 0 & 0 & 3/4 & \end{bmatrix}$$

Scratch

$$\begin{array}{cccc} 1 & 2 & 1 & 0 \\ -1 & -1/2 & 0 & 0 \end{array} \leftarrow -\tfrac{1}{2}R_1$$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$R_3 - \frac{2}{3}R_2 \to R_3 \quad m_{32} = \dfrac{1}{3/2} = \frac{2}{3}$

$R_4 - 0\,R_2 \to R_4 \quad m_{42} = \dfrac{0}{3/2} = 0$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Scratch

$0 \quad 1 \quad 2 \quad 1$

$0 \quad -1 \quad -2/3 \quad 0$

$R_4 - \frac{3}{4}R_3 \to R_4 \qquad m_{43} = \frac{1}{4/3} = \frac{3}{4}$

Scratch

$$\begin{matrix} 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & -3/4 \end{matrix}$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}}_{\text{this is } U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Example Continued...

Solve the linear system of equations

$$\begin{array}{ccccccccc} 2x_1 & + & x_2 & & & & & = & 7 \\ x_1 & + & 2x_2 & + & x_3 & & & = & 0 \\ & & x_2 & + & 2x_3 & + & x_4 & = & -4 \\ & & & & x_3 & + & 2x_4 & = & 0 \end{array}$$

We'll use $A = LU$ from the last example.

$$A\vec{x} = \vec{b} \implies LU\vec{x} = \vec{b}$$

Let $U\vec{x} = \vec{g}$ and solve $L\vec{g} = \vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 0 & 2/3 & 1 & 0 \\ 0 & 0 & 3/4 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \\ g_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$g_1 = 7$$

$$\frac{1}{2} g_1 + g_2 = 0$$

$$g_2 = -\frac{1}{2} g_1 = -\frac{7}{2}$$

$$\frac{2}{3} g_2 + g_3 = -4$$

$$\Rightarrow g_3 = -4 - \frac{2}{3} g_2 = -4 - \frac{2}{3}\left(-\frac{7}{2}\right) = -4 + \frac{7}{3} = \frac{-5}{3}$$

$$\frac{3}{4} g_3 + g_4 = 0 \Rightarrow g_4 = \frac{-3}{4} g_3 = \frac{-3}{4}\left(\frac{-5}{3}\right) = \frac{5}{4}$$

$U\vec{x} = \vec{g}$

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1 \\ 0 & 0 & 0 & 5/4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ -7/2 \\ -5/3 \\ 5/4 \end{bmatrix}$$

$$\frac{5}{4} x_4 = \frac{5}{4} \Rightarrow x_4 = 1$$

$$\frac{4}{3} x_3 + x_4 = -5/3 \Rightarrow \frac{4}{3} x_3 = \frac{-5}{3} - x_4 = \frac{-5}{3} - \frac{3}{3} = \frac{-8}{3}$$

$$x_3 = \frac{-8}{3} \cdot \frac{3}{4} = -2$$

$$\frac{3}{2}x_2 + x_3 = \frac{-7}{2}$$

$$\frac{3}{2}x_2 = \frac{-7}{2} - x_3 = \frac{-7}{2} + 2 = \frac{-3}{2}$$

$$x_2 = \frac{-3}{2} \cdot \frac{2}{3} = -1$$

$$2x_1 + x_2 = 7 \implies 2x_1 = 7 - x_2 = 7 + 1 = 8$$

$$x_1 = 4$$

$$\boxed{\text{The solution } \vec{x} = (4, -1, -2, 1)}$$

LU Decomposition: Pivoting

Not every matrix A can be written as $A = LU$ where L and U are lower and upper triangular. For example, show that the following equation is not solvable.

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \ell_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$$

This requires: $1u_{11} = 0$, $1u_{12} = 0$ $\Rightarrow u_{11} = u_{12} = 0$

$l_{21}u_{11} = 1$, $l_{21}u_{12} + 1u_{22} = 1$

$l_{21}u_{11} = l_{21} \cdot 0 = 0$ this can't be 1.

A doesn't not have a decomposition of this type.

LU Decomposition with Pivoting

Recall that at the k^{th} step in Gaussian elimination, the multiplier $m_{ij} = \frac{a_{ij}^{(k)}}{a_{jj}^{(k)}}$ which is undefined if $a_{jj}^{(k)} = 0$. For example, show that our process breaks down for the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 3 & 2 \end{bmatrix}$$

If we try

$$\begin{bmatrix} & & \\ 1 & & \\ 2 & & \end{bmatrix}$$

$R_2 - 1R_1 \to R_2 \quad m_{21} = \frac{2}{2} = 1$

$R_3 - 2R_1 \to R_3 \quad m_{31} = \frac{4}{2} = 2$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$$

We can't form m_{32} since

$$a_{22}^{(2)} = 0$$

$PA = LU$

The rows of a matrix A can be swapped by multiplying on the left with a permutation matrix P. P is obtained from the identity matrix by switching the desired rows.

For example: (a 3×3 example)

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{let} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The product PA results in switching rows 2 and 3 of A.

Compute PA

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -1 \\ 4 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix}$$

Show that PA has an LU decomposition

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 3 & 2 \\ 2 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} & & \\ 2 & & \\ 1 & & \end{bmatrix}$$

$R_2 - 2R_1 \to R_2 \quad m_{21} = \frac{4}{2} = 2$

$R_3 - R_1 \to R_3 \quad m_{31} = \frac{2}{2} = 1$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

This is U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$m_{32} = \frac{0}{1} = 0$

LU Decomposition Summary

To compute $A = LU$ or $PA = LU$

- If $a_{11} = 0$, swap rows to get a nonzero entry there, record the swap in a permutation matrix P being constructed.
- If a_{11} is nonzero, compute each m_{j1} and clear the first column of A below this entry.
- Store m_{j1} in the matrix L being constructed.
- If $a_{22}^{(2)} = 0$, swap rows to get a nonzero entry there, record the swap in the permutation matrix P being constructed. And swap the same rows in L being constructed.
- Otherwise, compute the multipliers m_{j2} and clear the second column of matrix.
- Store m_{j2} in the matrix L being constructed.
- Repeat until A is reduced to upper trianglar form. This is U.
- Add 1 to the diagonal entries of L.

Find an LU decomposition with pivoting if needed

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 6 & 2 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\begin{array}{c} P \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \qquad \begin{array}{c} L \\ \begin{bmatrix} & & \\ 2 & & \\ 4 & & \end{bmatrix} \end{array}$$

$R_2 - 2R_1 \to R_2 \quad m_{21} = 2$

$R_3 - 4R_1 \to R_3 \quad m_{31} = 4$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & 0 & -4 \\ 0 & -9 & -11 \end{bmatrix}$$

row swap is required

Scratch

$$\begin{array}{ccc} 2 & 6 & 2 \\ -2 & -6 & -6 \\ & & \\ 4 & 3 & 1 \\ -4 & -12 & -12 \end{array}$$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -9 & -11 \\ 0 & 0 & -4 \end{bmatrix} \quad M_{32} = 0 \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} & & \\ 4 & & \\ 2 & 0 & \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 3 & 3 \\ 0 & -9 & -11 \\ 0 & 0 & -4 \end{bmatrix}$$

Note $A \neq LU$ instead $PA = LU$

Solve the Linear System

$$\begin{array}{rcrcrcr} x_1 & + & 3x_2 & + & 3x_3 & = & 2 \\ 2x_1 & + & 6x_2 & + & 2x_3 & = & 5 \\ 4x_1 & + & 3x_2 & + & x_3 & = & 6 \end{array}$$

This is $A\vec{x} = \vec{b}$ from the last example.

We'll solve

$$PA\vec{x} = P\vec{b}$$

$$P\vec{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$$

Let $\vec{g} = U\vec{x}$

Solve $L\vec{g} = P\vec{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$$

$g_1 = 2$

$4g_1 + g_2 = 6 \Rightarrow g_2 = 6 - 4g_1 = 6 - 8 = -2$

$2g_1 + g_3 = 5 \Rightarrow g_3 = 5 - 2g_1 = 5 - 4 = 1$

$\vec{g} = (2, -2, 1)$

Solve $U\vec{x} = \vec{g}$

$$\begin{bmatrix} 1 & 3 & 3 \\ 0 & -9 & -11 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$$-4x_3 = 1 \Rightarrow x_3 = \frac{-1}{4}$$

$$-9x_2 - 11x_3 = -2 \Rightarrow -9x_2 = -2 + 11x_3 = -2 - \frac{11}{4}$$

$$-9x_2 = \frac{-19}{4} \Rightarrow x_2 = \frac{19}{36}$$

$$x_1 + 3x_2 + 3x_3 = 2$$

$$x_1 = 2 - 3x_2 - 3x_3 = 2 - \frac{57}{36} + \frac{3}{4}$$

$$= \frac{72 - 57 + 27}{36} = \frac{42}{36} = \frac{7}{6}$$

The solution

$$\vec{x} = \left(\frac{7}{6}, \frac{19}{36}, \frac{-1}{4} \right)$$