April 2 Math 2254H sec 015H Spring 2015

Section 11.6: Absolute Convergence: the Ratio & Root Tests

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(d) $$\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln n}$$

Ratio test: $a_n = \dfrac{(-1)^n}{n\ln(n)}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{(n+1)\ln(n+1)}\cdot\frac{n\ln(n)}{(-1)^n}\right|$$

$$= \lim_{n\to\infty}\frac{n\ln(n)}{(n+1)\ln(n+1)} = \lim_{n\to\infty}\frac{n}{n+1}\cdot\lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}^{*}$$

$$= 1$$

ratio test fails

$$\lim_{x\to\infty} \frac{\ln(x)}{\ln(x+1)} = \text{"}\frac{\infty}{\infty}\text{"} \quad \text{use l'H rule}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{1}{x+1}} = \lim_{x\to\infty} \frac{x+1}{x} = \lim_{x\to\infty}\left(1+\frac{1}{x}\right) = 1$$

Alt. series test: $b_n = \dfrac{1}{n\ln(n)}$

(ii) $\displaystyle \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n\ln(n)} = 0$

i) $\displaystyle b_{n+1} = \frac{1}{(n+1)\ln(n+1)} < \frac{1}{n\ln(n)} = b_n$

$\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln n}$ converges by the alt. series test.

To determine if it's conditional or absolute, consider

$$\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$

Int. test. $f(x) = \frac{1}{x \ln x}$

f is positive, continuous + decreasing on $[3, \infty)$

$$\int_3^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t \to \infty} \int_3^t \frac{dx}{x \ln x}$$

$$= \lim_{t \to \infty} \ln|\ln x| \Big|_3^t$$

$$= \lim_{t \to \infty} \left(\ln|\ln t| - \ln|\ln 3| \right) = \infty$$

divergent

So $\displaystyle\sum_{n=3}^{\infty} \left| \frac{(-1)^n}{n\ln n} \right|$ diverges

$$\boxed{\text{Hence } \sum_{n=3}^{\infty} \frac{(-1)^n}{n\ln(n)} \text{ converges conditionally.}}$$

$$\int \frac{dx}{x\ln x} = \int \frac{du}{u}$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

Examples

Use the ratio test to determine the values of t for which the series is guaranteed to be absolutely convergent.

$$\sum_{n=0}^{\infty} 4^n t^n$$

$a_n = 4^n t^n$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{4^{\cancel{n+1}}\, t^{\cancel{n+1}}}{\cancel{4^n}\,\cancel{t^n}}\right| = \lim_{n\to\infty} |4t|$$

$= 4|t|$ The series converges absolutely

if $4|t| < 1$

$|t| < \frac{1}{4}$

$$-\frac{1}{4} < t < \frac{1}{4}$$

So the series is guaranteed to converge absolutely if $-\frac{1}{4} < t < \frac{1}{4}$.

Ratio Test Failure

Apply the ratio test to the known **divergent** series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{n}{n+1} = 1$$

Ratio Test Failure

Apply the ratio test to the known **convergent** series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = 1^2 = 1$$

Theorem: The Root Test

Theorem (The Root Test): Let $\sum a_n$ be a series, and define the number L by

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Remark: In the case $L = 1$, the test truly fails as in the ratio test. The series may be absolutely convergent, conditionally convergent, or divergent.

Examples

Apply the root test to show that the series is absolutely convergent. (We get the same conclusion from noting that it is geometric.)

(a) $\displaystyle\sum_{n=0}^{\infty}\left(-\frac{2}{7}\right)^n$

Root test $\quad a_n = \left(\frac{-2}{7}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2}{7}\right)^n\right|}$$

$$= \lim_{n\to\infty} \sqrt[n]{\left|\frac{-2}{7}\right|^n} = \lim_{n\to\infty} \left(\frac{2}{7}\right)^{\frac{n}{n}} = \lim_{n\to\infty} \frac{2}{7} = \frac{2}{7}$$

$L = \frac{2}{7} < 1$ The series converges absolutely.

A Potentially Useful Result[1]

$$\lim_{x\to\infty} x^{\frac{1}{x}} = \text{"}\infty^0\text{"}$$

Show that $\displaystyle\lim_{x\to\infty} x^{1/x} = 1$

$$\lim_{x\to\infty} \ln\left(x^{\frac{1}{x}}\right) = \lim_{x\to\infty} \frac{1}{x}\ln(x) = \lim_{x\to\infty} \frac{\ln x}{x} = \text{"}\frac{\infty}{\infty}\text{"}$$

Use l'H rule

$$\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = \lim_{x\to\infty} \frac{1}{x} = 0$$

[1] We can conclude that

$$\lim_{n\to\infty} \sqrt[n]{n} = 1.$$

Hence $\lim_{x\to\infty} x^{\frac{1}{x}} = e^0 = 1.$

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(b) $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{2n-1}{n+3}\right)^n$

Root test $a_n = n\left(\frac{2n-1}{n+3}\right)^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{n\left(\frac{2n-1}{n+3}\right)^n}$$

$$= \lim_{n\to\infty} \sqrt[n]{n}\left(\frac{2n-1}{n+3}\right) = 1\cdot 2 = 2$$

$$L = 2 > 1$$

The series is divergent.

Section 11.7: Strategies for Testing Series

Potentially Useful Guidelines for Analyzing a Series $\sum a_n$

★ Does it have a specific *type*? (p-series, geometric, telescoping, alternating)

★ If you can readily see that $\lim_{n\to\infty} a_n \neq 0$, use the Divergence test.

★ If $a_n > 0$ and the function $f(n) = a_n$ looks like you can integrate it (i.e. $\int_1^\infty f(x)\,dx$ is manageable), try the integral test.

★ If it involves a rational function in n or a ratio of roots and powers of n, a direct or limit comparison test (comparing to a p-series) might be useful.

★ If it looks very similar to a geometric series, but is not quite a geometric series, a direct or limit comparison test to a geometric may be useful.

★ If it involves factorials or complicated products, the ratio test might lead to the necessary conclusion.

◇ Remember that the ratio test (when conclusive) determines absolute convergence. When using the alternating series test, if a series is found to be convergent remember to check for absolute convergence.