April 6 Math 2254H sec 015H Spring 2015

Section 11.7: Strategies for Testing Series

Special Series Types

- Geometric
- Telescoping
- *p*-Series
- Alternating

Tests

- Divergence (n^{th} term)
- Integral
- Direct & Limit Comparison
- Alternating Series
- Ratio test
- Root test

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{m=0}^{\infty} \frac{2^m}{3^m + 5^m}$ Ratio test

$$\lim_{m\to 0} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m\to\infty} \left| \frac{2^{\cancel{m}+1}}{3^{m+1}+5^{m+1}} \cdot \frac{3^m+5^m}{\cancel{2^m}} \right|$$

$$= \lim_{m\to\infty} 2 \left(\frac{3^m+5^m}{3^{m+1}+5^{m+1}} \right) \cdot \frac{\frac{1}{5^{m+1}}}{\frac{1}{5^{m+1}}}$$

$$= \lim_{m\to\infty} 2 \left(\frac{\frac{1}{5}\left(\frac{3}{5}\right)^m + \frac{1}{5}\left(\frac{5}{5}\right)^m}{\left(\frac{3}{5}\right)^{m+1} + 1} \right) = 2 \left(\frac{0+\frac{1}{5}}{0+1} \right) = \frac{2}{5}$$

$L = \frac{2}{5} < 1$ so the series

is absolutely convergent.

(b) $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k}{k!}$ Ratio test

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k\to\infty} \left| \frac{(-3)^{\cancel{k+1}}}{(k+1)!} \cdot \frac{k!}{\cancel{(-3)^k}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{-3\ k!}{k!\,(k+1)} \right| = \lim_{k\to\infty} \frac{3}{k+1} = 0$$

$$L = 0 < 1$$

The series is absolutely convergent.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{3n+2}{n-\sqrt{2}}$

Divergence Test:

$$\lim_{n\to\infty} \frac{3n+2}{n-\sqrt{2}} = \lim_{n\to\infty} \frac{3n+2}{n-\sqrt{2}} \cdot \frac{\frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{3+\frac{2}{n}}{1-\frac{\sqrt{2}}{n}} = \frac{3}{1} = 3 \neq 0$$

The series diverges.

(d) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{3n}{2n^2+3}$ Alt. Series Test. $b_n = \dfrac{3n}{2n^2+3}$

ii) $\displaystyle\lim_{n\to\infty}\frac{3n}{2n^2+3} = \lim_{n\to\infty}\frac{3n}{2n^2+3}\cdot\frac{\frac{1}{n^2}}{\frac{1}{n^2}}$

$\displaystyle = \lim_{n\to\infty}\frac{\frac{3}{n}}{2+3/n^2} = 0$

i) Need to show $b_{n+1} \leq b_n$

let $f(x) = \dfrac{3x}{2x^2+3}$, $f'(x) = \dfrac{3(2\cdot x^2+3) - 3x(4x)}{(2x^2+3)^2}$

$$f'(x) = \frac{-6x^2+9}{(2x^2+3)^2} = \frac{-6\left(x^2-\frac{3}{2}\right)}{(2x^2+3)^2}$$

$f'(x) < 0$ for $x > \sqrt{\frac{3}{2}}$ so $b_{n+1} \leq b_n$ for $n \geq 2$

The series converges by the alt. Series test.

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{(-1)^n\, 3n}{2n^2+3} \right| = \sum_{n=1}^{\infty} \frac{3n}{2n^2+3}$

Integral test: $f(x) = \dfrac{3x}{2x^2+3}$,

f is positive, continuous, and

decreasing (for $x > \sqrt{\frac{3}{2}}$).

$$\int_1^\infty \frac{3x}{2x^2+3}\,dx = \lim_{t\to\infty} \int_1^t \frac{3x}{2x^2+3}\,dx$$

$$= \lim_{t\to\infty} \frac{3}{4} \ln|2x^2+3| \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{3}{4} \ln|2t^2+3| - \frac{3}{4} \ln|2\cdot 1^2+3| \right)$$

$$= \infty$$

The integral, hence $\sum_{n=1}^\infty \frac{3n}{2n^2+3}$ diverges.

$$\int \frac{3x}{2x^2+3}\,dx$$

$$u = 2x^2+3$$

$$du = 4x\,dx$$

$$\frac{1}{4}\,du = x\,dx$$

$$= \frac{3}{4}\int \frac{du}{u} = \frac{3}{4}\ln|u| + C$$

$$= \frac{3}{4}\ln|2x^2+3| + C$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n \, 3n}{2n^2+3} \quad \text{is} \quad \text{conditionally convergent.}$$

Section 11.8: Power Series

Motivating Example: Let x be a variable (representing a real number). Show that the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

converges if $x = 3$ and diverges if $x = 7$.

When $x=3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(3-4)^n}{2n^2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{2n^2}$$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{2n^2} \right| = \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

a convergent p-series $p = 2 > 1$

So when $x=3$, the series is absolutely convergent.

If $x=7$, the series is $\sum_{n=1}^{\infty} \frac{(7-4)^n}{2n^2} = \sum_{n=1}^{\infty} \frac{3^n}{2n^2}$

Ratio test: $\lim_{n\to\infty} \left| \frac{3^{n+1}}{2(n+1)^2} \cdot \frac{2n^2}{3^n} \right| = \lim_{n\to\infty} 3\left(\frac{n}{n+1}\right)^2$

$= 3 > 1$

The series diverges when $x=7$.

Power Series

Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where the c_n's are (known) constants called the **coefficients**, x is a variable, and a is a (known) constant called the **center**.

For convenience, we set $(x-a)^0 = 1$ even in the case that $x = a$.

Remark: As the previous example suggests, a power series may be convergent for some values of x and divergent for others.

Example

Determine all value(s) of x for which the series converges.

we'll apply the ratio test w/ $x \neq 4$

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

$$\lim_{n\to\infty} \left| \frac{(x-4)^{\cancel{n}+1}}{\cancel{2}(n+1)^2} \cdot \frac{\cancel{2}n^2}{\cancel{(x-4)^n}} \right|$$

$$= \lim_{n\to\infty} |x-4| \frac{n^2}{(n+1)^2} = |x-4|$$

so $L = |x-4|$

The series converges absolutely if $|x-4| < 1$

$$|x-4| < 1 \Rightarrow -1 < x-4 < 1$$

$$\implies \quad 3 < x < 5$$

We know the series converges absolutely if $x=3$.

If $x=5$, the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(5-4)^n}{2n^2} = \sum_{n=1}^{\infty} \frac{1}{2n^2}$

which is absolutely convergent.

The series is absolutely convergent if

$$3 \le x \le 5 .$$

Note if $x > 5$ or $x < 3$

$L = |x - 4| > 1$

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=1}^{\infty} n!x^n$$

Ratio Test: for $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{(n+1)!\cdot \cancel{x^{n}}\, x}{n!\cdot \cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{n!}\cdot(n+1)\, x}{\cancel{n!}} \right|$$

$$= \lim_{n\to\infty} |x|(n+1) = \infty \qquad L = \infty$$

$L > 1$ for all $x \neq 0$

The series converges if $x=0$

and diverges for all

real $x \neq 0$.