April 7 Math 2254H sec 015H Spring 2015

Section 11.8: Power Series

Definition: A **power series** is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

where the c_n's are (known) constants called the **coefficients**, x is a variable, and a is a (known) constant called the **center.**

For convenience, we set $(x-a)^0 = 1$ even in the case that $x = a$.

Remark: A power series converges at its center to c_0. For other values of x is may or may not converge.

Examples

Example 1: We found that the series

$$\sum_{n=1}^{\infty} \frac{(x-4)^n}{2n^2}$$

coverges absolutely if $3 \leq x \leq 5$ and diverges if $x > 5$ or $x < 3$.

Example 2: We found that the series

$$\sum_{n=1}^{\infty} n! x^n$$

coverges at it center $x = 0$ and diverges if $x \neq 0$.

Example

Determine all value(s) of x for which the series converges.

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Ratio test: $x \neq 0$

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{2(n+1)}}{(2(n+1))!}\cdot\frac{(2n)!}{(-1)^n x^{2n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n}}\cdot x^2}{(2n+2)!}\cdot\frac{(2n)!}{\cancel{x^{2n}}}\right|$$

$$= \lim_{n\to\infty} \frac{x^2 \cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{x^2}{(2n+1)(2n+2)} = 0$$

$L = 0 < 1$

for all real x

This series converges for all real x.

Theorem on Power Series Convergence

Theorem: For the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are three possibilities:

(i) The series converges at the center $x = a$ and nowhere else.

(ii) The series converges for all real x; or

(iii) There exists a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

In the third case, R is called the **radius of convergence.**

Case (iii): Interval of Convergence

If there is a finite radius of convergence R, then the series converges for $|x - a| < R$. That is, for

$$a - R < x < a + R.$$

Behavior at the end points $x = a - R$ or $x = a + R$ varies from series to series. There are four possible cases. The **interval of convergence** may be any one of the following:

$$(i)\ a - R < x < a + R, \qquad (ii)\ a - R \le x < a + R,$$

$$(iiii)\ a - R < x \le a + R, \quad \text{or} \quad (iv)\ a - R \le x \le a + R.$$

Example

Determine the radius and interval of convergence of the power series. (If it converges for all real x, just set $R = \infty$.)

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}$$

Ratio test : $x \neq -1$

$$\lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{\cancel{n}+1}}{4^{\cancel{n}+1}} \cdot \frac{\cancel{4^n}}{n\cancel{(x+1)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{n+1}{4n} |x+1| = \lim_{n\to\infty} \frac{|x+1|}{4}\left(1+\frac{1}{n}\right) = \frac{|x+1|}{4}$$

The series converges absolutely if

$$\frac{|x+1|}{4} < 1 \implies |x+1| < 4$$

$R = 4$. $|x+1| < 4 \Rightarrow -4 < x+1 < 4$

$\Rightarrow -5 < x < 3$

Check the end points:

$x = -5$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{n(-5+1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n(-4)^n}{4^n}$

$$= \sum_{n=1}^{\infty} n\left(\frac{-4}{4}\right)^n = \sum_{n=1}^{\infty} (-1)^n n$$

This is divergent by the divergence test.

$x = 3$ $\quad \displaystyle\sum_{n=1}^{\infty} \frac{n(3+1)^n}{4^n} = \sum_{n=1}^{\infty} \frac{n4^n}{4^n} = \sum_{n=1}^{\infty} n$

Also divergent by the divergence test.

The radius of convergence is 4. And the interval is $(-5, 3)$.

Example

Determine the radius and interval of convergence of the power series. (If it converges for all real x, just set $R = \infty$.)

$$\sum_{n=1}^{\infty} \frac{2^n x^n}{\sqrt{n}}$$

Ratio test : $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{2^{\cancel{n}+1} x^{\cancel{n}+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{\cancel{2^n} \cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} 2|x| \sqrt{\frac{n}{n+1}} = 2|x|$$

The series converges absolutely if $2|x| < 1$

$$|x| < \frac{1}{2} \Rightarrow R = \frac{1}{2}$$

$$-\frac{1}{2} < x < \frac{1}{2}$$

End point check:

$x = -\frac{1}{2}$

$$\sum_{n=1}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{\left(2 \cdot -\frac{1}{2}\right)^n}{\sqrt{n}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}} \qquad b_n = \frac{1}{\sqrt{n}}$$

Alt Series test: $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$

$$b_{n+1} = \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} = b_n$$

This converges by the alt. Series test.

$x = \frac{1}{2}$ $\quad \sum_{n=1}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

a divergent p-series $p = \frac{1}{2} < 1$

The radius of convergence is $\frac{1}{2}$.

The interval is $\left[-\frac{1}{2}, \frac{1}{2}\right)$.

The convergence @ $x = -\frac{1}{2}$ is

conditional.

Example

Determine the radius and interval of convergence of the power series. (If it converges for all real x, just set $R = \infty$.)

$$\sum_{n=0}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n+1)}$$

Ratio test : $x \neq 0$

$$\lim_{n\to\infty} \left| \frac{x^{\cancel{n}+1}}{1\cdot 3\cdot 5\cdots(2(n+1)+1)} \cdot \frac{1\cdot 3\cdot 5\cdots(2n+1)}{\cancel{x^n}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x}{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)} \cdot \left(1\cdot 3\cdot 5\cdots(2n+1)\right) \right|$$

$$= \lim_{n\to\infty} \frac{|x|}{2n+3} = 0$$

$L = 0 < 1$

for all real

x

The radius is ∞. The interval is

$(-\infty, \infty)$.