April 10 Math 2306 sec. 53 Spring 2019

Section 15: Shift Theorems

Theorem (translation in s)
Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

Theorem (translation in t)
If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

Recall that the unit step function was defined as

$$\mathscr{U}(t-a) = \left\{\begin{array}{ll} 0, & 0 \le t < a \\ 1, & t \ge a \end{array}\right. \qquad \text{for } a > 0$$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

Since $g(t) = g((t+a)-a)$

Example: Find $\mathscr{L}\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\}$ $= e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\frac{\pi}{2})\}$

Note $\cos(t+\frac{\pi}{2}) = \cos t\cos\frac{\pi}{2} - \sin t\sin\frac{\pi}{2}$

$= \cos t\,(0) - \sin t\,(1)$

$= -\sin t$

So $\mathcal{L}\{\cos t\, \mathcal{U}(t-\frac{\pi}{2})\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t+\frac{\pi}{2})\}$

$$= e^{-\pi/2\, s}\mathcal{L}\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\mathcal{L}\{\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\left(\frac{1}{s^2+1^2}\right)$$

$$= -\frac{e^{-\pi/2\, s}}{s^2+1}$$

A Couple of Useful Results

The inverse form of this translation theorem is

(2) $\mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

we need to find $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$. Do a partial fraction decomp

$\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1} \Rightarrow 1 = A(s+1) + Bs$

set $s=0$ $\quad 1 = A(1) \quad A = 1$

$s=-1$ $\quad 1 = B(-1) \quad B = -1$

so
$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

← this is $f(t)$

so
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

↑ this is $f(t-2)\mathcal{U}(t-2)$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Integrate by parts

$u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -se^{-st}\,dt$ $\quad v = f(t)$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty -se^{-st}f(t)\,dt$$

$$= \left(0 - e^{0}f(0)\right) + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= s\mathscr{L}\{f(t)\} - f(0)$$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\begin{aligned}\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0)\\ &= s\left(sF(s) - f(0)\right) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0)\end{aligned}$$

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

We'll take the transform of both sides of the ODE

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\}.$$

Let $\mathscr{L}\{y(t)\} = Y(s)$ and $\mathscr{L}\{g(t)\} = G(s)$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{g\}$$

$$a\left(s^2Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

Let's isolate $Y(s)$.

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)$$

$$(as^2+bs+c)Y(s) - ay_0s - ay_1 - by_0 = G(s)$$

$$(as^2+bs+c)\,Y(s) = ay_0s + ay_1 + by_0 + G(s)$$

The ODE is $ay'' + by' + cy = g(t)$

so as^2+bs+c is the characteristic polynomial

$$Y(s) = \frac{ay_0 s + ay_1 + by_0}{as^2 + bs + c} + \frac{G(s)}{as^2 + bs + c}$$

The solution to the IVP is

$$y(t) = \mathcal{L}^{-1}\left\{ Y(s) \right\}$$

Solving IVPs

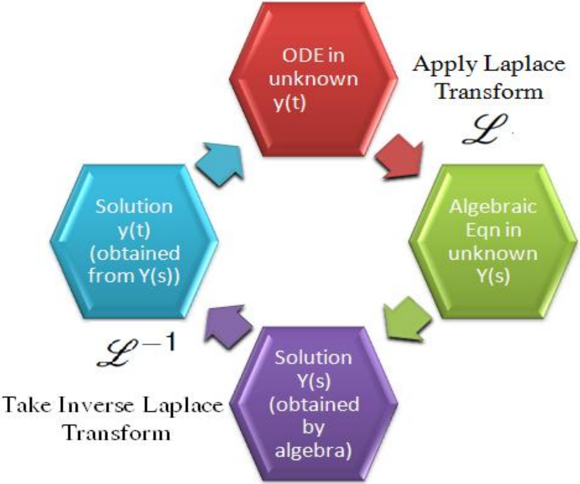

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**