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Math 2306 sec 58 Spring 2016

Section 13: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Find the Laplace transform of $f(t) = 1$

By definition $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1\, dt$

Note if $s=0$, $e^{-st} = e^{0} = 1$. In this case we have

$$\int_0^\infty 1\, dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t\Big|_0^b = \lim_{b\to\infty} (b-0) = \infty$$

Divergent. So 0 is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$, we have

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \frac{1}{-s} e^{-st} \Big|_0^\infty$$

This will diverge if $s < 0$

$$= \frac{1}{-s}\left(0 - e^{0}\right) \quad \text{for } s>0$$

$$= \frac{-1}{s}(-1) = \frac{1}{s}$$

So $\mathscr{L}\{1\} = \frac{1}{s}$ with domain $s>0$

* Recall $\lim_{x\to\infty} e^{x} = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$

Find the Laplace transform of $f(t) = t$

By definition $\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

If $s=0$, the integral is $\int_0^\infty t\, dt$ which diverges.

So 0 is not in the domain of $\mathscr{L}\{t\}$.

For $s \neq 0$

$$\mathscr{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

Int by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$= \frac{-1}{s} e^{-st}\, t \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

diverges if

$s < 0$

$$= \frac{-1}{s}(0-0) + \frac{1}{s}\int_0^\infty e^{-st}\,dt \qquad \text{for } s>0$$

$$= \frac{1}{s}\mathcal{L}\{1\}$$

$$= \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

So $\mathcal{L}\{t\} = \frac{1}{s^2}$ w/ domain $s>0$

* For $s>0$ $\lim_{t\to\infty} e^{-st}t = \lim_{t\to\infty}\frac{t}{e^{st}} = \text{"}\frac{\infty}{\infty}\text{"}$ Use l'Hospital's rule

$$= \lim_{t\to\infty}\frac{1}{se^{st}} = \frac{1}{\infty} = 0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\mathcal{L}\{f(t)\} = \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st}(2t)\,dt + \int_{10}^\infty e^{-st}\cdot 0\,dt$$

$$= 2\left[\frac{-1}{s} e^{-st}\cdot t \Big|_0^{10} - \int_0^{10} \frac{-1}{s} e^{-st}\,dt\right]$$

using the Int. by parts from the last example

$$= 2\left[\frac{-1}{s}\left(e^{-10s}\cdot 10 - e^{0}\cdot 0\right) + \frac{1}{s}\left(\frac{-1}{s}e^{-st}\Big|_0^{10}\right)\right]$$

$$= 2\left(\frac{-10}{s}e^{-10s} + \frac{1}{s}\left(\frac{-1}{s}e^{-10s} - \frac{-1}{s}e^{0}\right)\right]$$

$$= 2\left(\frac{-10}{s}e^{-10s} - \frac{1}{s^2}e^{-10s} + \frac{1}{s^2}\right)$$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} + \frac{2}{s^2} \quad \text{for } s \neq 0$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s} \quad \text{for } s \neq 0.$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace transform of

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s>0$$

(a) $f(t) = \cos(\pi t)$

Here $k=\pi$ so

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}, \quad s>0$$

Examples: Evaluate

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{for } s>0$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a} \quad \text{for } s>a$$

$$\mathscr{L}\{1\} = \frac{1}{s} \quad \text{for } s>0$$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathscr{L}\{2t^4 - e^{-5t} + 3\} = 2\mathscr{L}\{t^4\} - \mathscr{L}\{e^{-5t}\} + 3\mathscr{L}\{1\}$$

$$= 2\,\frac{4!}{s^5} - \frac{1}{s-(-5)} + 3\,\frac{1}{s}$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s} \quad \text{for } s>0$$

Examples: Evaluate

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$ Expand the square first

$$\mathscr{L}\{(2-t)^2\} = \mathscr{L}\{4 - 4t + t^2\}$$

$$= 4\mathscr{L}\{1\} - 4\mathscr{L}\{t\} + \mathscr{L}\{t^2\}$$

$$= 4\frac{1}{s} - 4\frac{1}{s^2} + \frac{2!}{s^3} \quad \text{for } s>0$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}, \quad s>0$$

Examples: Evaluate

Use the ID

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

(here $\theta = 5t$)

(d) $f(t) = \sin^2 5t$

$= \frac{1}{2} - \frac{1}{2}\cos(10t)$

$\mathcal{L}\{\sin^2 5t\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2}\cos(10t)\}$

$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$

$= \frac{1}{2}\frac{1}{s} - \frac{1}{2}\frac{s}{s^2+10^2}$ for $s>0$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+100}$, $s>0$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order* c provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

(No vertical asymptotes)

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

$f(t) = e^{t^2}$ doesn't have a Laplace transform because it grows faster than e^{ct} for every real number c.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

Note that

$$\frac{1}{s^7} = \frac{6!}{s^7}\cdot\frac{1}{6!} = \frac{1}{720}\,\frac{6!}{s^7}$$

$$= \mathscr{L}^{-1}\left\{\frac{1}{720}\,\frac{6!}{s^7}\right\}$$

$$= \frac{1}{720}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{720}\,t^6$$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\} = \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}\right\}$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \mathscr{L}^{-1}\left\{\frac{1}{3}\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

$* \ \mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

We require a partial fraction decomp on $\dfrac{s-8}{s(s-2)}$

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Clear fractions
mult. by $s(s-2)$

$$s-8 = A(s-2) + Bs$$

let $s=2$ $\quad 2-8 = A(0) + B(2)$

$$-6 = 2B \implies B = -3$$

Let $s=0$ $\quad 0-8 = A(-2)+B(0)$

$$-8 = -2A \Rightarrow A=4$$

So $\quad \dfrac{s-8}{s(s-2)} = \dfrac{4}{s} - \dfrac{3}{s-2}$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s(s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3\cdot e^{2t} = 4-3e^{2t}$$