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Math 2306 sec 59 Spring 2016

Section 13: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Find the Laplace transform of $f(t) = 1$

By definition $\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$

If $s=0$, $e^{-st} = e^0 = 1$. In this case we have

$$\int_0^\infty 1 \, dt = \lim_{b\to\infty} \int_0^b dt = \lim_{b\to\infty} t \Big|_0^b = \lim_{b\to\infty} (b-0) = \infty$$

The integral diverges, so zero is not in the domain of $\mathcal{L}\{1\}$.

For $s \neq 0$

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \, dt = \frac{1}{-s} e^{-st} \Big|_0^\infty$$

Diverges if $s<0$

$$= \frac{1}{-s}\left(0 - e^{0}\right) \quad \text{for} \quad s > 0$$

$$= \frac{1}{-s}(-1) = \frac{1}{s}$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ w/ domain $s > 0$.

Recall: $\lim_{x\to\infty} e^{x} = \infty$ and $\lim_{x\to\infty} e^{-x} = 0$

Find the Laplace transform of $f(t) = t$

By definition $\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$

For $s=0$, the integral is $\int_0^\infty t\, dt$ which diverges.

Again, zero is not in the domain of $\mathcal{L}\{t\}$.

For $s \neq 0$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

Int. by parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$= \frac{-1}{s} e^{-st}\, t \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

$$= \frac{-1}{s}(0-0) + \frac{1}{s}\int_0^\infty e^{-st}\,dt \qquad \text{for} \quad s>0$$

$$= 0 + \frac{1}{s}\mathcal{L}\{1\} \qquad (\text{from before})$$

$$= \frac{1}{s}\cdot\frac{1}{s} = \frac{1}{s^2}$$

So $\mathcal{L}\{t\} = \frac{1}{s^2}$, with domain $s>0$

* Note for $s>0$ $\lim_{t\to\infty} e^{-st}t = \lim_{t\to\infty}\frac{t}{e^{st}} = \text{"}\frac{\infty}{\infty}\text{"}$ Use l'Hospital's rule

$$= \lim_{t\to\infty}\frac{1}{se^{st}} = 0$$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

, By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\mathcal{L}\{f(t)\} = \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} (2t)\,dt + \int_{10}^\infty e^{-st} \cdot 0\,dt$$

$$= 2\int_0^{10} e^{-st} t\,dt$$

If $s=0$, we get

$$\mathcal{L}\{f(t)\} = 2\int_0^{10} t\,dt = t^2\Big|_0^{10} = 100$$

If $s \neq 0$

$$\mathcal{L}\{f(t)\} = 2\int_0^{10} e^{-st}\, t\,dt$$

Using the Int by parts work from before

$$= 2\left[\frac{-1}{s}e^{-st}\,t\,\Big|_0^{10} + \frac{1}{s}\int_0^{10} e^{-st}\,dt\right.$$

$$= 2\left[\frac{-1}{s}e^{-st}\,t\,\Big|_0^{10} + \frac{1}{s}\left[\frac{-1}{s}e^{-st}\right|_0^{10}\right.$$

$$= 2\left[\frac{-1}{s}e^{-10s}\cdot 10 - \frac{-1}{s}e^{0}\cdot 0 - \frac{1}{s^2}\left(e^{-10s} - e^{0}\right)\right]$$

$$= 2\left(\frac{-10}{s}e^{-10s} - \frac{1}{s^2}e^{-10s} + \frac{1}{s^2}\right)$$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} + \frac{2}{s^2}$$

so $$\mathcal{L}\{f(t)\} = \begin{cases} 100, & s=0 \\ \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}, & s \neq 0 \end{cases}$$

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \dots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace transform of

(a) $f(t) = \cos(\pi t)$

Use

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s>0$$

Here, $k=\pi$

So

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}, \quad s>0$$

Examples: Evaluate

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$$

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, \quad s>a$$

$$\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0$$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\frac{4!}{s^{4+1}} - \frac{1}{s-(-5)} + 3\frac{1}{s}$$

$$s>0 \qquad s>-5 \qquad s>0$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}, \quad s>0$$

Examples: Evaluate

(c) $f(t) = (2-t)^2$ Expand the square first

$$f(t) = 4 - 4t + t^2$$

$$\mathcal{L}\{(2-t)^2\} = \mathcal{L}\{4 - 4t + t^2\}$$

$$= 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4 \cdot \frac{1}{s} - 4 \cdot \frac{1}{s^2} + \frac{2!}{s^{2+1}}, \quad s > 0$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Examples: Evaluate

Use the ID

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$

take $\theta = 5t$

(d) $f(t) = \sin^2 5t$

$f(t) = \frac{1}{2} - \frac{1}{2}\cos(10t)$

$$\mathcal{L}\{\sin^2(5t)\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(10t)\right\}$$

$$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$$

$$= \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+10^2}, \quad s>0$$

$$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+100}, \quad s>0$$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

f can go to ∞, but no faster than an exponential e^{ct}

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

(no vertical asymptotes)

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0, \infty)$ and of exponential order c for some $c > 0$, then f has a Laplace transform for $s > c$.

$f(t) = e^{t^2}$ doesn't have a Laplace transform.

$f \to \infty$ faster than e^{ct} for any number c.