April 14 Math 2306 sec 58 Spring 2016

Section 14: Inverse Laplace Transforms

If $\mathscr{L}\{f(t)\} = F(s)$, we call $F(s)$ the Laplace transform of $f(t)$.

We'll call $f(t)$ an inverse Laplace transform of $F(s)$, and write

$$\mathscr{L}^{-1}\{F(s)\} = f(t)$$

A Table of Laplace Transforms

Some basic results include:

- $\mathscr{L}\{\alpha f(t)+\beta g(t)\}=\alpha F(s)+\beta G(s)$
- $\mathscr{L}\{1\}=\frac{1}{s}, \quad s>0$
- $\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}}, \quad s>0$ for $n=1,2,\ldots$
- $\mathscr{L}\{e^{at}\}=\frac{1}{s-a}, \quad s>a$
- $\mathscr{L}\{\cos kt\}=\frac{s}{s^2+k^2}, \quad s>0$
- $\mathscr{L}\{\sin kt\}=\frac{k}{s^2+k^2}, \quad s>0$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s)+\beta G(s)\}=\alpha f(t)+\beta g(t)$$

Find the Transform or Inverse Transform

(a) $\mathscr{L}\{\sin(t)\cos(t)\}$

Recall
$\sin(2\theta) = 2\sin\theta\cos\theta$

so
$\frac{1}{2}\sin(2t) = \sin t\cos t$

$= \mathscr{L}\left\{\frac{1}{2}\sin(2t)\right\}$

$= \frac{1}{2}\mathscr{L}\{\sin(2t)\}$

$= \frac{1}{2}\,\frac{2}{s^2+2^2} = \frac{1}{s^2+4}$

(b) $\mathscr{L}^{-1}\left\{\dfrac{1}{s^2-1}\right\}$

Use partial fraction decomp on $\dfrac{1}{s^2-1}$

multiply by $(s-1)(s+1)$

$$(s-1)(s+1)\quad \frac{1}{s^2-1} = \frac{1}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1}$$

$$1 = A(s+1) + B(s-1)$$

$$1 = As + A + Bs - B$$

$$\underline{\underline{0}}s + \underline{\underline{1}} = \underline{\underline{(A+B)}}s + \underline{\underline{(A-B)}}$$

$$A+B=0$$

$$A-B=1$$

sum

$$2A = 1$$

$$A = \frac{1}{2}, \quad B = -A = \frac{-1}{2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} - \frac{\frac{1}{2}}{s+1}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$$

$*\ s+1$

$= s-(-1)$

$$*\ \mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

Note that

$$\mathscr{L}\{e^t t^2\} = \int_0^\infty e^{-st} e^t t^2 \, dt$$

$$= \int_0^\infty e^{-(s-1)t} t^2 \, dt$$

This is $F(s) = \mathscr{L}\{t^2\}$ but with s replaced w/ $s-1$.

Properties of exponentials

$$e^{-st} \cdot e^{t} = e^{-st+t}$$

$$= e^{(-s+1)t}$$

$$= e^{-(s-1)t}$$

So If $F(s) = \mathcal{L}\{t^2\}$, then $\mathcal{L}\{e^t t^2\} = F(s-1)$

From the table

$$F(s) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$$

So $$F(s-1) = \frac{2}{(s-1)^3}$$ i.e. $\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = e^t t^2$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Everywhere s appears, it's replaced

w/ $s-a$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is an irreducible quadratic

$b^2-4ac = 2^2-4\cdot1\cdot2 = 4-8 < 0$

Complete the square

$$s^2+2s+2 = s^2+2s+1-1+2$$
$$= (s^2+2s+1)+1$$
$$= (s+1)^2+1$$

So

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1}$$

we need all s terms to be replaced w/ $s+1$

$$= \frac{s+1-1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}$$

$$\mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

here $s+1$
$= s-(-1)$

$$= e^{-t}\cos t - e^{-t}\sin t$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\frac{1+3s-s^2}{s(s-1)^2}\right\}$

Do partial Fraction Decomp.

$s(s-1)^2 \quad \frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$

mult by $s(s-1)^2$

$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$

We'll finish this during the next class.