April 14 Math 2306 sec 59 Spring 2016

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets {} **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

Note $\frac{1}{s^7} = \frac{1}{6!}\frac{6!}{s^7} = \frac{1}{720}\frac{6!}{s^7}$

$= \mathscr{L}^{-1}\left\{\frac{1}{720}\frac{6!}{s^7}\right\}$

$= \frac{1}{720}\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{720}t^6$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\frac{s+1}{s^2+9}\right\}$

Note

$$\frac{s+1}{s^2+9} = \frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}$$

$$= \frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos(kt)$$

$$\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin(kt)$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

we'll do partial fraction decomp on $\dfrac{s-8}{s(s-2)}$

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

Clear fractions multiply both sides by $s(s-2)$

$$\frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$s-8 = A(s-2) + Bs$$

Let $s=0$ $\quad 0-8 = A(0-2)$

$-8 = -2A \Rightarrow A = 4$

$s=2$ $\quad 2-8 = A(2-2) + B\cdot 2$

$-6 = 2B \Rightarrow B = -3$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4\cdot 1 - 3\cdot e^{2t}$$

$$= 4 - 3e^{2t}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

Properties of Exponentials

$$e^{-st} \cdot e^{t} = e^{-st+t}$$

$$= e^{(-s+1)t}$$

$$= e^{-(s-1)t}$$

Consider

$$\mathscr{L}\{e^t t^2\} = \int_0^\infty e^{-st} e^t t^2 \, dt$$

$$= \int_0^\infty e^{-(s-1)t} t^2 \, dt$$

This looks like $\mathscr{L}\{t^2\}$ except s is replaced w/ $s-1$. If $F(s) = \mathscr{L}\{t^2\}$, this is $F(s-1)$.

$$\mathcal{L}\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = F(s)$$

So $$F(s-1) = \frac{2}{(s-1)^3}$$

That is, $$\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\} = e^t t^2$$

Theorem (translation in s)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a).$$

For example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathscr{L}\{e^{at} t^n\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2} \implies \mathscr{L}\{e^{at}\cos(kt)\} = \frac{s-a}{(s-a)^2+k^2}.$$

Every where s is replaced by s-a

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible

$b^2-4ac = 2^2-4\cdot1\cdot2 = 4-8 < 0$

We'll complete the square

$$s^2+2s+2 = s^2+2s+1-1+2$$
$$= (s^2+2s+1)+1 \quad = (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

$*$ $\mathcal{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$ and $\mathcal{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

$s+1 = s-(-1)$