April 15 MATH 1112 sec. 54 Spring 2019

Section 7.5: Trigonometric Equations

A General Observation: When solving more complicated trigonometric equations, we will try to rewrite the problem in the form of **one or more** equations that look like

$$\text{One Trig Function} \quad = \quad \text{One Number}$$

We typically determine solution(s) in one period, and then extend those solutions if required.

Example

Find all possible solutions of the equation $1 + \cos\theta = 2\sin^2\theta$.

We can try to rewrite the equation to have only sine or only cosine. Note $\sin^2\theta + \cos^2\theta = 1$

The equation is

$$1 + \cos\theta = 2(1 - \cos^2\theta)$$
$$= 2 - 2\cos^2\theta$$

$$2\cos^2\theta - 2 + 1 + \cos\theta = 0$$

$$2\cos^2\theta + \cos\theta - 1 = 0$$

Looks like

$$2u^2 + u - 1 = 0$$
$$(2u - 1)(u + 1) = 0$$

The trig equation is

$$(2\cos\theta - 1)(\cos\theta + 1) = 0$$

By the zero product property

$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

We solved the first equation on 4/10/19 and found

$\theta = \frac{\pi}{3} + 2\pi k$ or

$\theta = \frac{5\pi}{3} + 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$

In one revolution (θ in $[0, 2\pi)$), there is one solution to $\cos\theta = -1$

$$\theta = \pi$$

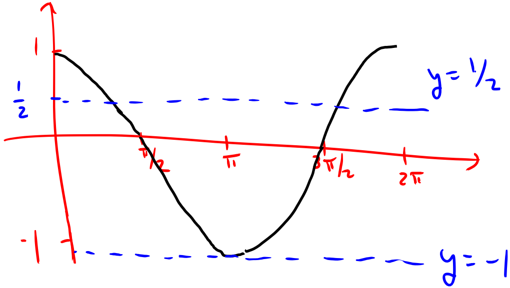
All solutions are $\theta = \pi + 2\pi k$ for k an integer.

All solutions to $1 + \cos\theta = 2\sin^2\theta$ are

$$\theta = \frac{\pi}{3} + 2\pi k, \text{ or } \theta = \frac{5\pi}{3} + 2\pi k, \text{ or } \theta = \pi + 2\pi k$$

for $k = 0, \pm 1, \pm 2, \ldots$

Consider $y = \cos x$ on $[0, 2\pi)$



Solutions on an Indicated Interval

Find all solutions of the equation $\cos^2(2x) = \frac{1}{4}$ on the interval $0 \le x < 2\pi$.

One solution process is to consider $\cos^2\theta = \frac{1}{4}$, solve this, then let $\theta = 2x$. If $0 \le x < 2\pi$ and $\theta = 2x$, then

$$0 \cdot 2 \le 2 \cdot x < 2 \cdot (2\pi)$$

$$0 \le 2x < 4\pi$$

so $0 \le \theta < 4\pi$ (2 full rotations)

$$\cos^2\theta = \frac{1}{4} \Rightarrow \cos\theta = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

$\cos\theta = \frac{1}{2}$ or $\cos\theta = \frac{-1}{2}$

Recall that if the reference angle is $\frac{\pi}{3}$ then the cosine is $\frac{1}{2}$ or $\frac{-1}{2}$.

In 2 full rotations

$$\cos\theta = \frac{1}{2} \text{ if } \theta = \frac{\pi}{3},\ \theta = \frac{5\pi}{3},\ \theta = \frac{7\pi}{3} \text{ or}$$

$$\theta = \frac{11\pi}{3}$$

In 2 full rotations

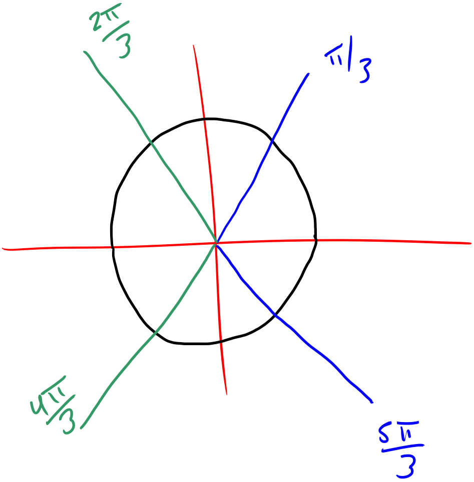
$$\cos\theta = \frac{-1}{2} \quad \theta = \frac{2\pi}{3},\ \theta = \frac{4\pi}{3},\ \theta = \frac{8\pi}{3},\ \theta = \frac{10\pi}{3}$$

Now, $\theta = 2x$ so $x = \dfrac{\theta}{2}$

$$x = \frac{\pi/3}{2}, \frac{5\pi/3}{2}, \frac{\frac{7\pi}{3}}{2}, \frac{11\pi/3}{2} = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

or

$$x = \frac{2\pi/3}{2}, \frac{4\pi/3}{2}, \frac{8\pi/3}{2}, \frac{10\pi/3}{2} = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$



2π/3
π/3
4π/3
5π/3

Squaring & Superfluous Solutions

If both sides of an equation are squared, this may introduce extraneous solutions. To weed out such *apparent (but false)* solutions, check all answers.

Example: Find all solutions of $\tan(x) - 1 = \sec(x)$ on the interval $0 \le x < 2\pi$.

One ID is $\tan^2 x + 1 = \sec^2 x$. Let's square both sides of the equation.

$$(\tan x - 1)^2 = (\sec x)^2$$

$$\tan^2 x - 2\tan x + 1 = \sec^2 x$$

Now use the Pythagorean ID

$$\tan^2 x - 2\tan x + 1 = \tan^2 x + 1$$

Subtract $\tan^2 x + 1$

$$-2\tan x = 0$$

$$\tan x = 0$$

In $[0, 2\pi)$, this is true when

$x = 0$ or when $x = \pi$.

We need to check if these solve the original equation.

$$\tan x - 1 = \sec x$$

Check $x = 0$

$$\tan 0 - 1 \stackrel{?}{=} \sec 0$$

$$0 - 1 \stackrel{?}{=} 1$$

$$-1 = 1 \quad \text{false!}$$

$x = 0$ is not a solution

Check $x = \pi$

$$\tan \pi - 1 \stackrel{?}{=} \sec \pi$$

$$0 - 1 \stackrel{?}{=} -1$$

$$-1 = -1 \quad \text{true}$$

The only solution is $x = \pi$.

Using Inverse Trigonometric Functions

Find the solutions of the equation $4\sin\theta = 1$ on $[0, 2\pi)$. Express answers exactly in terms of the inverse sine.

Isolate $\sin\theta$: $\sin\theta = \frac{1}{4}$

$\frac{1}{4}$ is not a "nice" sine value.

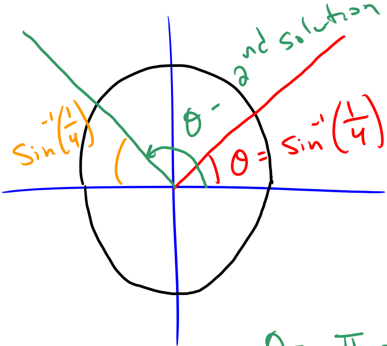
$\sin\theta = \frac{1}{4}$ should have one quadrant I answer and on quadrant II answer (since $\frac{1}{4}$ is positive)

Recall : the range of $\sin^{-1} x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

One solution to $\sin\theta = \frac{1}{4}$ is

$$\theta = \sin^{-1}\left(\frac{1}{4}\right)$$

That's a 1st quadrant solution.



The 2nd solution is $\theta = \pi - \sin^{-1}\left(\frac{1}{4}\right)$

The solutions to $\sin\theta = \frac{1}{4}$ are

$$\theta = \sin^{-1}\left(\frac{1}{4}\right) \quad \text{and} \quad \theta = \pi - \sin^{-1}\left(\frac{1}{4}\right)$$