April 15 Math 2306 sec. 53 Spring 2019

Section 16: Laplace Transforms of Derivatives and IVPs

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Solving IVPs

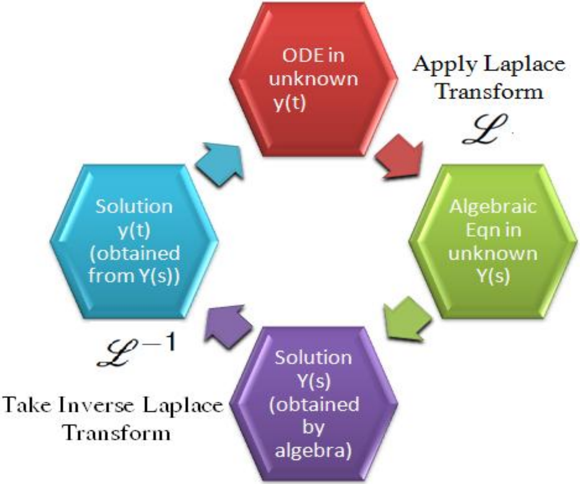


Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Let $Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y' + 3y\} = \mathcal{L}\{2t\}$

$\mathcal{L}\{y'\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$

$sY(s) - y(0) + 3Y(s) = 2\left(\dfrac{1!}{s^2}\right)$

$(s+3)Y(s) - 2 = \dfrac{2}{s^2}$

$(s+3)Y(s) = \dfrac{2}{s^2} + 2 \quad \Rightarrow \quad Y(s) = \dfrac{2}{s^2(s+3)} + \dfrac{2}{s+3}$

We'll decompose $\dfrac{2}{s^2(s+3)}$

$$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$ Clear fractions

$$2 = As(s+3) + B(s+3) + Cs^2$$
$$= A(s^2+3s) + B(s+3) + Cs^2$$
$$2 = (A+C)s^2 + (3A+B)s + 3B$$

Match coefficients

$$A + C = 0 \qquad \Rightarrow C = -A = \frac{2}{9}$$
$$3A + B = 0 \qquad \Rightarrow A = \frac{-B}{3} = \frac{-2}{9}$$
$$3B = 2 \qquad \Rightarrow B = \frac{2}{3}$$

So $$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{2/9}{s+3} + \frac{2}{s+3}$$

$$= \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}$$

The solution to the IVP $y(t) = \mathcal{L}^{-1}\{Y(s)\}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}\right\}$$

$$= \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y(t) = \frac{-2}{9} + \frac{2}{3}t + \frac{20}{9}e^{-3t}$$

Solve the IVP using the Laplace Transform

$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$ Let $\mathcal{L}\{y(t)\} = Y(s)$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ and $\mathcal{L}\{t\} = \frac{1}{s^2}$

$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{(s+2)^2}$

$s^2 \cdot Y(s) - s \cdot 1 - 0 + 4sY(s) - 4 \cdot 1 + 4Y(s) = \frac{1}{(s+2)^2}$

$$(s^2+4s+4)Y(s) - s - 4 = \frac{1}{(s+2)^2}$$

$$(s^2+4s+4)Y(s) = \frac{1}{(s+2)^2} + s + 4$$

$$Y(s) = \frac{1}{(s^2+4s+4)(s+2)^2} + \frac{s+4}{s^2+4s+4}$$

$s^2+4s+4 = (s+2)^2$ and $s+4 = s+2+2$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+2+2}{(s+2)^2}$$

$$= \frac{1}{(s+2)^4} + \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

From before, $\mathcal{L}\{te^{-2t}\} = \frac{1}{(s+2)^2}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\} = \frac{1}{3!}\mathcal{L}^{-1}\left\{\frac{3!}{s^4}\right\} = \frac{1}{3!}t^3$$

so $\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} = \frac{1}{3!}t^3 e^{-2t}$

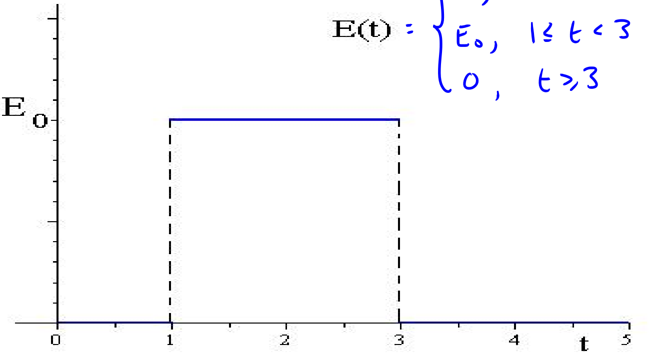
$$y(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$\boxed{y(t) = \frac{1}{3!}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}}$$

Solve the IVP

An LR-series circuit has inductance $L = 1$h, resistance $R = 10\Omega$, and applied force $E(t)$ whose graph is given below. If the initial current $i(0) = 0$, find the current $i(t)$ in the circuit.



$$E(t) = \begin{cases} 0, & 0 \le t < 1 \\ E_0, & 1 \le t < 3 \\ 0, & t \ge 3 \end{cases}$$

The ODE is $L\frac{di}{dt} + Ri = E$

LR Circuit Example

Writing E in terms of unit steps

$$E(t) = 0 - 0\,\mathcal{U}(t-1) + E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3) + 0\,\mathcal{U}(t-3)$$

The IVP is

$$\frac{di}{dt} + 10i = E_0\,\mathcal{U}(t-1) - E_0\,\mathcal{U}(t-3), \quad i(0) = 0$$

Let $\mathcal{L}\{i(t)\} = I(s)$

$$\mathcal{L}\left\{\frac{di}{dt}\right\} + 10\,\mathcal{L}\{i\} = E_0\,\mathcal{L}\{\mathcal{U}(t-1)\} - E_0\,\mathcal{L}\{\mathcal{U}(t-3)\}$$

$$sI(s) - \underset{0}{\underset{\shortparallel}{i(0)}} + 10\,I(s) = E_0\frac{e^{-s}}{s} - \frac{E_0\,e^{-3s}}{s}$$

$$(s+10)\,I(s) = E_0\frac{e^{-s}}{s} - E_0\frac{e^{-3s}}{s}$$

$$I(s) = E_0\frac{e^{-s}}{s(s+10)} - E_0\frac{e^{-3s}}{s(s+10)}$$

We'll decompose $\dfrac{1}{s(s+10)}$

$$\frac{1}{s(s+10)} = \frac{A}{s} + \frac{B}{s+10}$$

$$\Rightarrow \quad 1 = A(s+10) + Bs$$

Set $s=0$ $\quad 1 = 10A \quad A = \frac{1}{10}$

$s=-10 \quad 1 = -10B \quad B = \frac{-1}{10}$

So

$$I(s) = \frac{E_0}{10} e^{-s}\left(\frac{1}{s} - \frac{1}{s+10}\right) - \frac{E_0}{10} e^{-3s}\left(\frac{1}{s} - \frac{1}{s+10}\right)$$

$$\mathcal{L}^{-1}\left\{e^{-as} F(s)\right\} = f(t-a)\,\mathcal{U}(t-a)$$

where $f(t) = \mathcal{L}^{-1}\left\{F(s)\right\}$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}-\frac{1}{s+10}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+10}\right\}$$

$$= 1 - e^{-10t}$$

The current $i(t) = \mathcal{L}^{-1}\{I(s)\}$

$$i(t) = \mathcal{L}^{-1}\left\{\frac{E_0}{10} e^{-s}\left(\frac{1}{s}-\frac{1}{s+10}\right)\right\} - \mathcal{L}^{-1}\left\{\frac{E_0}{10} e^{-3s}\left(\frac{1}{s}-\frac{1}{s+10}\right)\right\}$$

$$= \frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-s}\left(\frac{1}{s}-\frac{1}{s+10}\right)\right\} - \frac{E_0}{10}\mathcal{L}^{-1}\left\{e^{-3s}\left(\frac{1}{s}-\frac{1}{s+10}\right)\right\}$$

$$i(t) = \frac{E_0}{10}\left(1 - e^{-10(t-1)}\right)u(t-1) - \frac{E_0}{10}\left(1 - e^{-10(t-3)}\right)u(t-3)$$