April 19 Math 2306 sec. 53 Spring 2019

Section 17: Fourier Series: Trigonometric Series

The **Fourier series** of the function f defined on $(-\pi, \pi)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right).$$

Where

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx, \\ a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx, \quad \text{and} \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx \end{aligned}$$

Example

Find the Fourier series of the piecewise defined function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 \le x < \pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} f(x)\,dx + \frac{1}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\,dx = \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\cos(nx)\,dx \overset{=0}{} + \frac{1}{\pi}\int_{0}^{\pi} x\cos(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin(nx) + \frac{1}{n^2}\cos(nx)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\sin(n\pi) + \frac{1}{n^2}\cos(n\pi) - \frac{0}{n}\sin(0) - \frac{1}{n^2}\cos(0)\right]$$

Note $\sin(n\pi) = 0$ for all $n = \pm 1, \pm 2, \ldots$

$$\cos(n\pi) = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases} = (-1)^n$$

$$a_n = \frac{1}{\pi}\left[0 + \frac{1}{n^2}(-1)^n - 0 - \frac{1}{n^2}\right] = \frac{(-1)^n - 1}{n^2\pi}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx = \frac{1}{\pi}\int_{-\pi}^{0} 0\cdot\sin(nx)\,dx + \frac{1}{\pi}\int_{0}^{\pi} x\sin(nx)\,dx$$

$$= \frac{1}{\pi}\left[\frac{-x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right|_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\pi}{n}\cos(n\pi) + \frac{1}{n^2}\sin(n\pi) - \left(\frac{-0}{n}\cos(0) + \frac{1}{n^2}\sin 0\right)\right]$$

$$= \frac{-1}{n}(-1)^n = \frac{(-1)^{n+1}}{n}$$

We found $a_0 = \frac{\pi}{2}$, $a_n = \frac{(-1)^n - 1}{n^2\pi}$, and $b_n = \frac{(-1)^{n+1}}{n}$

So

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2\pi}\cos(nx) + \frac{(-1)^{n+1}}{n}\sin(nx)\right]$$

Some Observations

The following appear regularly when computing Fourier coefficients:

$$\cos(n\pi) = (-1)^n$$

$$\sin(n\pi) = 0$$

for all integers n.

An Orthogonal Set of Functions on $[-p, p]$

This set can be generalized by using a simple change of variables $t = \frac{\pi x}{p}$ to obtain the orthogonal set on $[-p, p]$

$$\left\{1, \cos\frac{n\pi x}{p}, \sin\frac{m\pi x}{p} \,|\, n, m = \pm 1, \pm 2, \ldots\right\}$$

There are many interesting and useful orthogonal sets of functions (on appropriate intervals). What follows is readily extended to other such (infinite) sets.

Fourier Series on an interval $(-p, p)$

The set of functions $\{1, \cos\left(\frac{n\pi x}{p}\right), \sin\left(\frac{m\pi x}{p}\right) | n, m \geq 1\}$ is orthogonal on $[-p, p]$. Moreover, we have the properties

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)\,dx = 0 \quad \text{and} \quad \int_{-p}^{p} \sin\left(\frac{m\pi x}{p}\right)\,dx = 0 \;\; \text{for all} \;\; n, m \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\,dx = 0 \quad \text{for all} \quad m, n \geq 1,$$

$$\int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right) \cos\left(\frac{m\pi x}{p}\right)\,dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases},$$

$$\int_{-p}^{p} \sin\left(\frac{n\pi x}{p}\right) \sin\left(\frac{m\pi x}{p}\right)\,dx = \begin{cases} 0, & m \neq n \\ p, & n = m \end{cases}.$$

Fourier Series on an interval $(-p, p)$

The orthogonality relations provide for an expansion of a function f defined on $(-p, p)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right)\right)$$

where

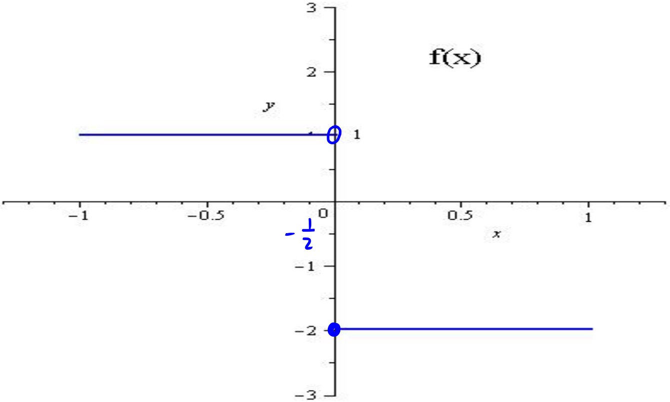
$$\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\,dx, \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\left(\frac{n\pi x}{p}\right)\,dx, \quad \text{and} \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\left(\frac{n\pi x}{p}\right)\,dx
\end{aligned}$$

Find the Fourier series of f

f is defined on $(-p,p)$

$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \le x < 1 \end{cases}$$

$p=1$ so $\frac{n\pi x}{p} = \frac{n\pi x}{1} = n\pi x$



$$f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1} \int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} -2\,dx$$

$$= \left[x\right|_{-1}^{0} + \left[-2x\right|_{0}^{1} = (0-(-1)) + (-2-0) = -1$$

$$a_n = \frac{1}{1} \int_{-1}^{1} f(x) \cos(n\pi x)\,dx = \int_{-1}^{0} \cos(n\pi x)\,dx + \int_{0}^{1} -2\cos(n\pi x)\,dx$$

$$= \frac{1}{n\pi}\sin(n\pi x)\Big|_{-1}^{0} - \frac{2}{n\pi}\sin(n\pi x)\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\sin(0) - \sin(-n\pi) - 2\left(\sin(n\pi) - \sin(0)\right)\right]$$

$$= 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{0}\sin(n\pi x)\,dx + \int_{0}^{1} -2\sin(n\pi x)\,dx$$

$$= \frac{-1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0} + \frac{2}{n\pi}\cos(n\pi x)\Big|_{0}^{1}$$

$$= \frac{-1}{n\pi}\left[\cos(0) - \cos(-n\pi)\right] + \frac{2}{n\pi}\left[\cos(n\pi) - \cos(0)\right]$$

$$= \frac{-1}{n\pi} + \frac{1}{n\pi}(-1)^n + \frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}$$

$$= \frac{3}{n\pi}(-1)^n - \frac{3}{n\pi} \quad = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

We have $a_0 = -1$, $a_n = 0$ $n \geq 1$ and

$$b_n = \frac{3}{n\pi}\left((-1)^n - 1\right)$$

So $$f(x) = \frac{-1}{2} + \sum_{n=1}^{\infty} \frac{3}{n\pi}\left((-1)^n - 1\right)\sin(n\pi x)$$

Convergence?

The last example gave the series

$$f(x) = -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{3((-1)^n - 1)}{n\pi} \sin(n\pi x).$$

This example raises an interesting question: The function f is not continuous on the interval $(-1, 1)$. However, each term in the Fourier series, and any partial sum thereof, is obviously continuous. This raises questions about properties (e.g. continuity) of the series. More to the point, we may ask: *what is the connection between f and its Fourier series at the point of discontinuity?*

This is the convergence issue mentioned earlier.

Convergence of the Series

Theorem: If f is continuous at x_0 in $(-p, p)$, then the series converges to $f(x_0)$ at that point. If f has a jump discontinuity at the point x_0 in $(-p, p)$, then the series **converges in the mean** to the average value

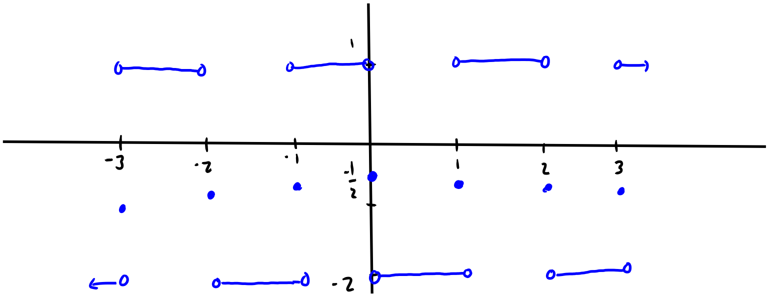
$$\frac{f(x_0-) + f(x_0+)}{2} \stackrel{\text{def}}{=} \frac{1}{2}\left(\lim_{x \to x_0^-} f(x) + \lim_{x \to x_0^+} f(x)\right)$$

at that point.

Periodic Extension:
The series is also defined for x outside of the original domain $(-p, p)$. The extension to all real numbers is $2p$-periodic.

Periodic Extension & Convergence in the Mean

For $f(x) = \begin{cases} 1, & -1 < x < 0 \\ -2, & 0 \leq x < 1 \end{cases}$, plot the series over the interval $(-3, 3)$. Determine the values to which the series converges when $x = 0, \frac{1}{2}, 1$ and 2.



x	0	$\frac{1}{2}$	1	2
Series	$\frac{-1}{2}$	-2	$-1/2$	$-1/2$