April 19 Math 2306 sec 58 Spring 2016

Section 15: Shift Theorems

Theorem (translation in s)
Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

We were doing a partial fraction decomposition and wrote

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2} \quad \Longrightarrow$$

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs = A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2 + 3s + 1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$A+B=-1$$

$$-2A-B+C=3$$

$$A=1$$

$$B = -1 - A = -1 - 1 = -2$$

$$C = 3 + B + 2A = 3 - 2 + 2(1) = 3$$

$$\mathscr{L}^{-1}\left\{ \frac{-s^2 + 3s + 1}{s(s-1)^2} \right\} = \mathscr{L}^{-1}\left\{ \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2} \right\}$$

$$= \mathscr{L}^{-1}\left\{ \frac{1}{s} \right\} - 2\mathscr{L}^{-1}\left\{ \frac{1}{s-1} \right\} + 3\mathscr{L}^{-1}\left\{ \frac{1}{(s-1)^2} \right\}$$

$$= 1 - 2e^t + 3te^t$$

$$* \quad \mathcal{L}\{t\} = \frac{1!}{s^2} = \frac{1}{s^2}$$

and $\frac{1}{(s-1)^2}$ is $\frac{1}{s^2}$ w/ s

replaced with $s-1$.

so $$\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = t \cdot e^{1t}$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside Step function

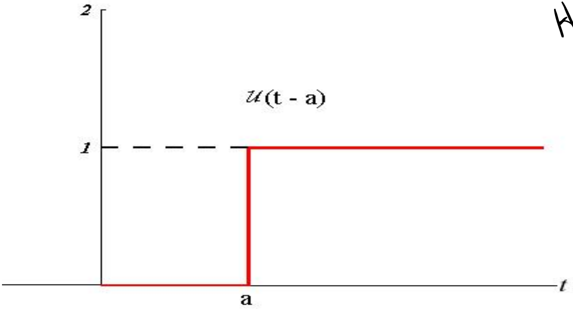

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We need to consider the cases $0 \le t < a$ and $t \geq a$.

If $0 \le t < a$ then $\mathscr{U}(t-a) = 0$. So for $0 \le t < a$

$f(t) = g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t)$ as is required

If $t \geq a$, then $\mathcal{U}(t-a) = 1$. So for $t \geq a$

$$f(t) = g(t) - g(t) \cdot 1 + h(t) \cdot 1 = h(t)$$

again as required.

So

$$\begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a).$$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

$$f(t) = e^t - e^t\mathscr{U}(t-2) + t^2\mathscr{U}(t-2) - t^2\mathscr{U}(t-5) + 2t\,\mathscr{U}(t-5)$$

turn off e^t

turn on t^2

turn off t^2

turn on $2t$

Let's verify

If $0 \leq t < 2$ then $t < 5$ so $\mathscr{U}(t-2) = 0$ and $\mathscr{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t \quad \checkmark$$

If $2 \leq t < 5$ then $\mathscr{U}(t-2) = 1$ and $\mathscr{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2 \quad \checkmark$$

If $t \geq 5$ then $t > 2$ so $\mathscr{U}(t-2) = 1$ and $\mathscr{U}(t-5) = 1$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t \quad \checkmark$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

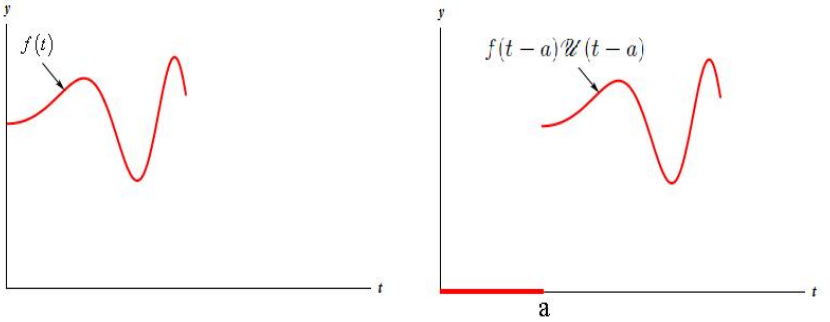

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s)=\mathscr{L}\{f(t)\}$ and $a>0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\}=e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\}=\frac{e^{-as}}{s}.$$

here the $f(t)=1$ so $f(t-a)=1$ $\mathscr{L}\{1\}=\frac{1}{s}$

As another example,

$$\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}} \implies \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\}=\frac{n!e^{-as}}{s^{n+1}}.$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1+t)\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{1}{s^2}\cdot e^{-1s} = \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$*$ If $f(t) = t$ then $f(t-1) = t-1$ $\quad \mathscr{L}\{t\} = \frac{1}{s^2}$

A Couple of Useful Results

Another formulation of this translation theorem is

(1) $\mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$

Note $g(t) = g((t+a)-a)$

Example: Find $\mathscr{L}\left\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$ Here $a = \frac{\pi}{2}$

$$= e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$$

Use $\cos(A+B) = \cos A\cos B - \sin A\sin B$

So $\cos(t+\pi/2) = \cos t\cos\pi/2 - \sin t\sin\pi/2 = -\sin t$

$$\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi/2)\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t+\pi/2)\}$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s}\frac{1}{s^2+1^2}$$

$$= \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

Step 1: Ignore the exponential.

What we need is $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

Partial Fraction: $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$$1 = A(s+1) + Bs$$

Set $s=0$ $\quad 1 = A(0+1) + B\cdot 0 = A \Rightarrow A = 1$

$s=-1$ $\quad 1 = A(-1+1) + B(-1) = -B \Rightarrow B = -1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

Step 2

find the inverse transform

So our $f(t) = 1 - e^{-t}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$$

Step 3:

Use the

Shifting

theorem

$$= \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

$$= \mathcal{U}(t-2) - e^{-(t-2)}\mathcal{U}(t-2)$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= e^{-st} f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st} f(t))\,dt$$

$$= (0 - e^{0} f(0)) + s\int_0^\infty e^{-st} f(t)\,dt$$

for $s > c$

$$= -f(0) + s\mathcal{L}\{f(t)\}$$

If $\quad F(s) = \mathcal{L}\{f(t)\}$

Then $\quad \mathcal{L}\{f'(t)\} = sF(s) - f(0)$

Transforms of Derivatives

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\mathscr{L}\{f'(t)\} = sF(s) - f(0)$. We can use this relationship recursively to obtain Laplace transforms for higher derivatives of f.

For example

$$\begin{aligned}
\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(sF(s) - f(0)\right) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}$$

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Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\left\{y(t)\right\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

Let $\mathcal{L}\{y\} = Y(s)$ and $\mathcal{L}\{g\} = G(s)$

$$\mathcal{L}\{ay'' + by' + cy\} = \mathcal{L}\{g(t)\}$$

$$a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y\} = G(s)$$

$$a\left(s^2 Y(s) - sy(0) - y'(0)\right) + b\left(sY(s) - y(0)\right) + cY(s) = G(s)$$

we'll finish on thursday.