April 19 Math 2306 sec 59 Spring 2016

Section 15: Shift Theorems

Theorem (translation in s)
Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Do a partial fraction decomp on $\dfrac{1+3s-s^2}{s(s-1)^2}$

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions mult. by $s(s-1)^2$

$$-s^2+3s+1 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$\left.\begin{aligned} A+B &= -1 \\ -2A-B+C &= 3 \\ A &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} B &= -1-A = -1-1 = -2 \\ C &= 3+B+2A = 3-2+2\cdot 1 = 3 \end{aligned}$$

$$\frac{-s^2+3s+1}{s(s-1)^2} = \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{-s^2+3s+1}{s(s-1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + 3\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} \quad *$$

$$= 1 - 2e^t + 3te^t$$

Note $\mathcal{L}\{t\} = \frac{1}{s^2}$ and $\frac{1}{(s-1)^2}$ is this function

with s replaced by $s-1$.

So $\mathscr{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = t \cdot e^{1t} = te^t$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Heaviside Step function

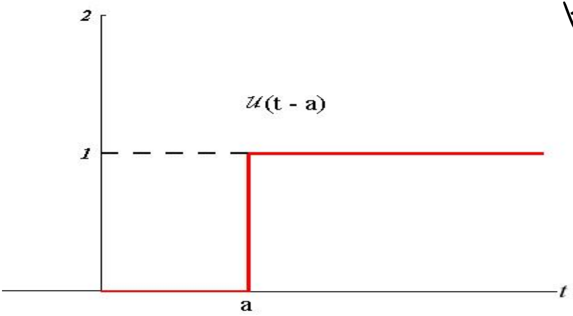

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \quad g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We need to examine the right side when $0 \le t < a$ and when $t \ge a$.

For $0 \le t < a$, $\mathscr{U}(t-a) = 0$

Then the right side is

$g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t)$ as required.

When $t \geq a$, $\mathcal{U}(t-a) = 1$. Then the right side is

$$g(t) - g(t)\cdot 1 + h(t)\cdot 1 = h(t)$$

again as required

Hence

$$\begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)\mathcal{U}(t-a) + h(t)\mathcal{U}(t-a).$$

turn g on at $t=0$

turn off g at $t=a$

turn h on at $t=a$

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t)=\begin{cases} e^t, & 0\le t<2\\ t^2, & 2\le t<5\\ 2t & t\ge 5\end{cases}$$

$$f(t)=e^t-e^t\mathscr{U}(t-2)+t^2\mathscr{U}(t-2)-t^2\mathscr{U}(t-5)+2t\,\mathscr{U}(t-5)$$

turn off e^t @ $t=2$

turn on t^2 @ $t=2$

turn off t^2 @ $t=5$

turn on $2t$ @ $t=5$

Let's verify that this is correct

For $0 \le t < 2$, $t < 5$ so $\mathcal{U}(t-2) = 0$ and $\mathcal{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t \checkmark$$

For $2 \le t < 5$, $\mathcal{U}(t-2) = 1$ $\mathcal{U}(t-5) = 0$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2 \checkmark$$

For $t \geq 5$, note $t > 2$ $\mathcal{U}(t-2) = 1$ $\mathcal{U}(t-5) = 1$

$$f(t) = e^t \cdot e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t \quad \checkmark$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

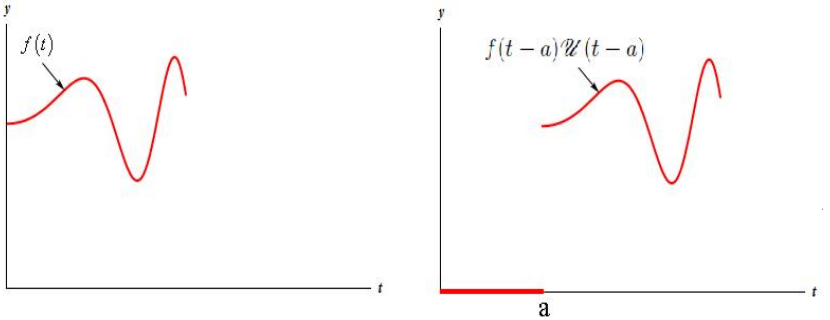

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as}F(s).$$

the $f(t)$ here is so that
$f(t)=1$
$f(t-a)=1$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & t \ge 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

$$= 1 + (-1+t)\,\mathcal{U}(t-1)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

We want to recognize

$$(t-1)\mathcal{U}(t-1) \quad \text{as} \quad h(t-1)\mathcal{U}(t-1)$$

for some function $h(t)$

Here $h(t) = t$ note this

gives $h(t-1) = t-1$ as required

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{1}{s^2} \cdot e^{-1s}$$

$$= \frac{1}{s} + \frac{e^{-s}}{s^2}$$

$$* \ \mathscr{L}\{t\} = \frac{1}{s^2}$$

A Couple of Useful Results

Another formulation of this translation theorem is

$$(1) \quad \mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

$$g(t) = g((t+a)-a)$$

Example: Find $\mathscr{L}\left\{\cos t\,\mathscr{U}\left(t-\frac{\pi}{2}\right)\right\}$ $\qquad a = \pi/2$

$$= e^{-\frac{\pi}{2}s}\mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$$

Use $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$$\cos\left(t+\frac{\pi}{2}\right) = \cos t\cos\frac{\pi}{2} - \sin t\sin \pi/2 \quad = -\sin t$$

So $\mathscr{L}\{\cos t\,\mathscr{U}(t-\pi/2)\} = e^{-\frac{\pi}{2}s}\mathscr{L}\{\cos(t+\pi/2)\}$

$$= e^{-\frac{\pi}{2}s}\mathscr{L}\{-\sin t\}$$

$$= e^{-\frac{\pi}{2}s}\left(\frac{-1}{s^2+1^2}\right)$$

$$= \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

e^{2t}

$e^{2(t+1)}$

$= e^{2t+2}$

$= e^{2}e^{2t}$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$

1st step: Ignore the exponential

We need $\mathscr{L}^{-1}\left\{\dfrac{1}{s(s+1)}\right\}$

Partial fractions $\dfrac{1}{s(s+1)} = \dfrac{A}{s} + \dfrac{B}{s+1}$

$$1 = A(s+1) + Bs$$

Set $s=0$ $\quad 1 = A(0+1) + B\cdot 0 = A \qquad A=1$

$s=-1 \quad 1 = A(-1+1) + B(-1) = -B \qquad B=-1$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t}$$

Step 2:

Find the inverse transform

$f(t)$

Here $f(t) = 1 - e^{-t}$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = f(t-2)\mathcal{U}(t-2)$$

Step 3: Use the translation theorem

$$= \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$

$$= \mathcal{U}(t-2) - e^{-(t-2)}\mathcal{U}(t-2)$$

Section 16: Laplace Transforms of Derivatives and IVPs

Suppose f has a Laplace transform and that f is differentiable on $[0,\infty)$. Obtain an expression for the Laplace tranform of $f'(t)$. (Assume f is of exponential order c for some c.)

$$\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty e^{-st} f'(t)\,dt$$

Int. by parts

$u = e^{-st}$ $\quad du = -se^{-st}\,dt$

$v = f(t)$ $\quad dv = f'(t)\,dt$

$$= e^{-st}f(t)\Big|_0^\infty - \int_0^\infty (-se^{-st})f(t)\,dt$$

for $s > c$

$$= \left(0 - e^{0}f(0)\right) + s\int_0^\infty e^{-st}f(t)\,dt$$

$$= -f(0) + s\,\mathcal{L}\{f(t)\}$$

If $\mathcal{L}\{f(t)\} = F(s)$ then

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$$