April 21 Math 2306 sec 58 Spring 2016

Section 16: Laplace Transforms of Derivatives and IVPs

If $\mathscr{L}\{f(t)\} = F(s)$, we have $\quad \mathscr{L}\{f'(t)\} = sF(s) - f(0)$.

Extending this

$$
\begin{aligned}
\mathscr{L}\left\{f''(t)\right\} &= s\mathscr{L}\left\{f'(t)\right\} - f'(0) \\
&= s\left(sF(s) - f(0)\right) - f'(0) \\
&= s^2F(s) - sf(0) - f'(0)
\end{aligned}
$$

Similarly $\mathscr{L}\{f'''(t)\} = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$.

Transforms of Derivatives

For $y = y(t)$ defined on $[0, \infty)$ having derivatives y', y'' and so forth, if

$$\mathscr{L}\{y(t)\} = Y(s),$$

then

$$\mathscr{L}\left\{\frac{dy}{dt}\right\} = sY(s) - y(0),$$

$$\mathscr{L}\left\{\frac{d^2y}{dt^2}\right\} = s^2Y(s) - sy(0) - y'(0),$$

$$\vdots$$

$$\mathscr{L}\left\{\frac{d^ny}{dt^n}\right\} = s^nY(s) - s^{n-1}y(0) - s^{n-2}y'(0) - \cdots - y^{(n-1)}(0).$$

Differential Equation

For constants a, b, and c, take the Laplace transform of both sides of the equation

$$ay'' + by' + cy = g(t), \quad y(0) = y_0, \quad y'(0) = y_1$$

We started this by letting $Y(s) = \mathscr{L}\{y(t)\}$ and $G(s) = \mathscr{L}\{g(t)\}$. We took the transform of both sides of the equation, using the linearity and new derivative properties

$$\mathscr{L}\{ay'' + by' + cy\} = \mathscr{L}\{g(t)\} \quad \Longrightarrow$$

$$a\mathscr{L}\{y''\} + b\mathscr{L}\{y'\} + c\mathscr{L}\{y\} = \mathscr{L}\{g(t)\} \quad \Longrightarrow$$

$$a\underbrace{\left(s^2Y(s) - sy(0) - y'(0)\right)}_{\mathscr{L}\{y''\}} + b\underbrace{(sY(s) - y(0))}_{\mathscr{L}\{y'\}} + c\underbrace{Y(s)}_{\mathscr{L}\{y\}} = G(s)$$

We'll Isolate $Y(s)$

$$as^2Y(s) - asy(0) - ay'(0) + bsY(s) - by(0) + cY(s) = G(s)$$

$$(as^2+bs+c)Y(s) - asy(0) - ay'(0) - by(0) = G(s)$$

$$(as^2+bs+c)Y(s) - ay_0s - ay_1 - by_0 = G(s)$$

$$(as^2+bs+c)Y(s) = ay_0s + ay_1 + by_0 + G(s)$$

$$Y(s) = \frac{ay_0s + ay_1 + by_0}{as^2+bs+c} + \frac{G(s)}{as^2+bs+c}$$

This is the transform $\mathscr{L}\{y(t)\}$. To solve the IVP, we take the inverse transform $y(t) = \mathscr{L}^{-1}\{Y(s)\}$

Solving IVPs

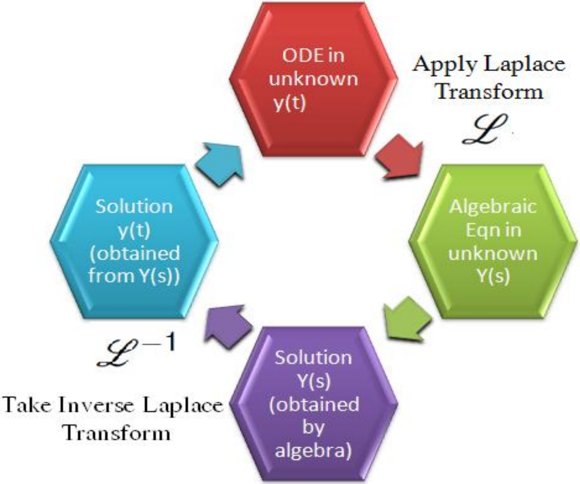

Figure: We use the Laplace transform to turn our DE into an algebraic equation. Solve this transformed equation, and then transform back.

General Form

We get

$$Y(s) = \frac{Q(s)}{P(s)} + \frac{G(s)}{P(s)}$$

where Q is a polynomial with coefficients determined by the initial conditions, G is the Laplace transform of $g(t)$ and P is the **characteristic polynomial** of the original equation.

$$\mathscr{L}^{-1}\left\{\frac{Q(s)}{P(s)}\right\}$$ is called the **zero input response**,

and

$$\mathscr{L}^{-1}\left\{\frac{G(s)}{P(s)}\right\}$$ is called the **zero state response.**

Solve the IVP using the Laplace Transform

(a) $\dfrac{dy}{dt} + 3y = 2t \quad y(0) = 2$

Take $\mathcal{L}\{\}$ of the ODE.

Let $Y(s) = \mathcal{L}\{y\}$

$$\mathcal{L}\left\{\frac{dy}{dt} + 3y\right\} = \mathcal{L}\{2t\}$$

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y\} = 2\mathcal{L}\{t\}$$

Isolate $Y(s)$

$$sY(s) - y(0) + 3Y(s) = 2\frac{1}{s^2}$$

$$(s+3)Y(s) - 2 = \frac{2}{s^2} \implies (s+3)Y(s) = \frac{2}{s^2} + 2$$

$$Y(s) = \frac{2}{s^2(s+3)} + \frac{2}{s+3}$$

Now do what's needed to find $y = \mathcal{L}^{-1}\{Y(s)\}$.

Partial fractions:

$s^2(s+3)$ $$\frac{2}{s^2(s+3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+3}$$ $s^2(s+3)$

$$2 = As(s+3) + B(s+3) + Cs^2$$

$$= A(s^2+3s) + B(s+3) + Cs^2$$

$$0s^2 + 0s + 2 = (A+C)s^2 + (3A+B)s + 3B$$

$A + C = 0 \Rightarrow C = -A$

$3A + B = 0 \Rightarrow B = -3A$ $\quad A = \frac{-1}{3}B = \frac{-1}{3}\cdot\frac{2}{3} = \frac{-2}{9} \quad C = -A = \frac{2}{9}$

$3B = 2 \Rightarrow B = \frac{2}{3}$

$$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{2/9}{s+3} + \frac{2}{s+3}$$

$$2 + \frac{2}{9} = \frac{20}{9}$$

$$Y(s) = \frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}$$

Find y

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{-2/9}{s} + \frac{2/3}{s^2} + \frac{20/9}{s+3}\right\}$$

$$y(t) = \frac{-2}{9}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{2}{3}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \frac{20}{9}\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\}$$

$$y(t) = \frac{-2}{9} \cdot 1 + \frac{2}{3} t + \frac{20}{9} e^{-3t}$$

$$\boxed{y(t) = \frac{-2}{9} + \frac{2}{3} t + \frac{20}{9} e^{-3t}}$$

Solve the IVP using the Laplace Transform

$y'' + 4y' + 4y = te^{-2t} \quad y(0) = 1, y'(0) = 0$

$Y(s) = \mathcal{L}\{y(t)\}$

$\mathcal{L}\{y'' + 4y' + 4y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 4\mathcal{L}\{y\} = \mathcal{L}\{te^{-2t}\}$

$\mathcal{L}\{t\} = \frac{1}{s^2}$

$s^2Y(s) - sy(0) - y'(0) + 4\left(sY(s) - y(0)\right) + 4Y(s) = \frac{1}{(s-(-2))^2}$

$(s^2 + 4s + 4)Y(s) - sy(0) - y'(0) - 4y(0) = \frac{1}{(s+2)^2}$

$(s^2 + 4s + 4)Y(s) - s \cdot 1 - 4 \cdot 1 = \frac{1}{(s+2)^2}$

$$(s^2+4s+4)Y(s) = \frac{1}{(s+2)^2} + s+4$$

$$Y(s) = \frac{1}{(s+2)^2(s^2+4s+4)} + \frac{s+4}{s^2+4s+4}$$

$$s^2+4s+4 = (s+2)^2$$

$$Y(s) = \frac{1}{(s+2)^2(s+2)^2} + \frac{s+4}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+2+2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{s+2}{(s+2)^2} + \frac{2}{(s+2)^2}$$

$$Y(s) = \frac{1}{(s+2)^4} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

Note $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3!}\,\frac{3!}{s^4}\right\} = \frac{1}{3!}\,t^3$

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{(s+2)^4}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{(s+2)^2}\right\}$$

$$= \frac{1}{3!}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}$$

$$\boxed{y(t) = \frac{1}{6}t^3 e^{-2t} + e^{-2t} + 2te^{-2t}}$$

Shift Thm:

If $F(s) = \mathcal{L}\{f(t)\}$ then

$$F(s-a) = \mathcal{L}\{e^{at} f(t)\}$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \quad \text{so} \quad \mathcal{L}\{t^3\} = \frac{3!}{s^4}$$