

Section 17: Fourier Series: Trigonometric Series

Find the Fourier Series for $f(x) = x$, $-1 < x < 1$

$$p = 1 \quad \frac{n\pi x}{p} = n\pi x$$

$$a_0 = \frac{1}{1} \int_{-1}^1 f(x) dx = \int_{-1}^1 x dx = \left. \frac{x^2}{2} \right|_{-1}^1 = \frac{1^2}{2} - \frac{(-1)^2}{2} = 0$$

$$a_n = \frac{1}{1} \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_{-1}^1 x \cos(n\pi x) dx$$

$$= \left[\frac{x}{n\pi} \sin(n\pi x) + \frac{1}{n^2 \pi^2} \cos(n\pi x) \right]_{-1}^1$$

$$= \frac{1}{n^2 \pi^2} \cos(n\pi) - \frac{1}{n^2 \pi^2} \cos(-n\pi) = 0$$

$a_n = 0$ for $n = 0, 1, 2, 3, \dots$

$$b_n = \frac{1}{i} \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_{-1}^1 x \sin(n\pi x) dx$$

$$= \left[\frac{-x}{n\pi} \cos(n\pi x) + \frac{1}{n^2 \pi^2} \sin(n\pi x) \right]_{-1}^1$$

$$= \frac{-1}{n\pi} \cos(n\pi) - \frac{-(-1)}{n\pi} \cos(-n\pi)$$

$$= \frac{-1}{n\pi} (-1)^n - \frac{1}{n\pi} (-1)^n = \frac{-2}{n\pi} (-1)^n = \frac{2}{n\pi} (-1)^{n+1}$$

$$a_0 = a_n = 0 \text{ for } n \geq 1 \text{ and}$$

$$b_n = \frac{2}{n\pi} (-1)^{n+1}$$

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} (-1)^{n+1} \sin(n\pi x)$$

Symmetry

For $f(x) = x$, $-1 < x < 1$

$$f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin(n\pi x)$$

Observation: f is an odd function. It is not surprising then that there are no nonzero constant or cosine terms (which have even symmetry) in the Fourier series for f .

The following plots show f , f plotted along with some partial sums of the series, and f along with a partial sum of its series extended outside of the original domain $(-1, 1)$.

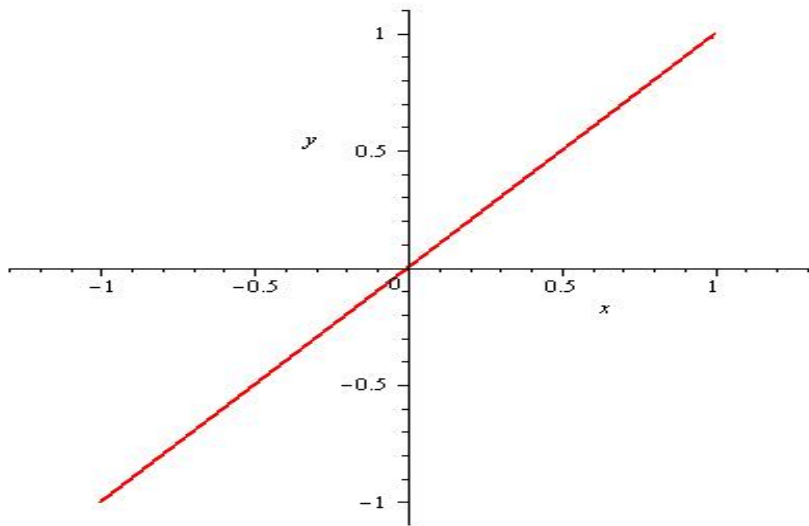


Figure: Plot of $f(x) = x$ for $-1 < x < 1$

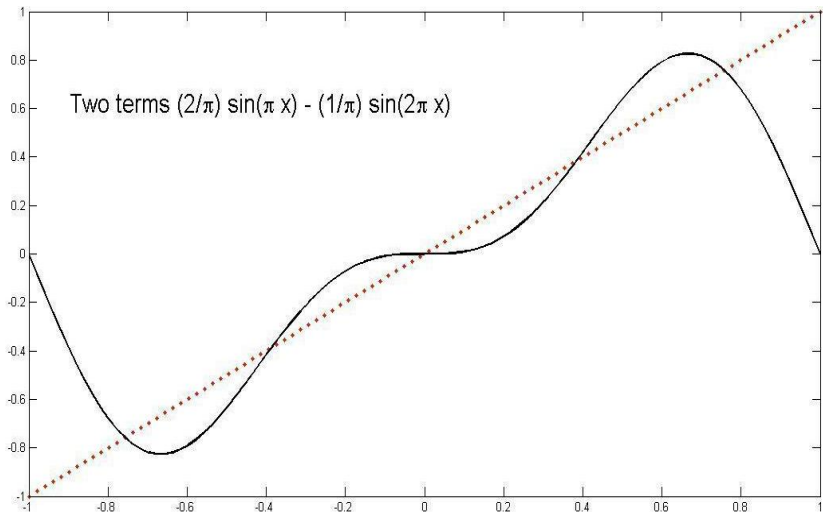


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with two terms of the Fourier series.

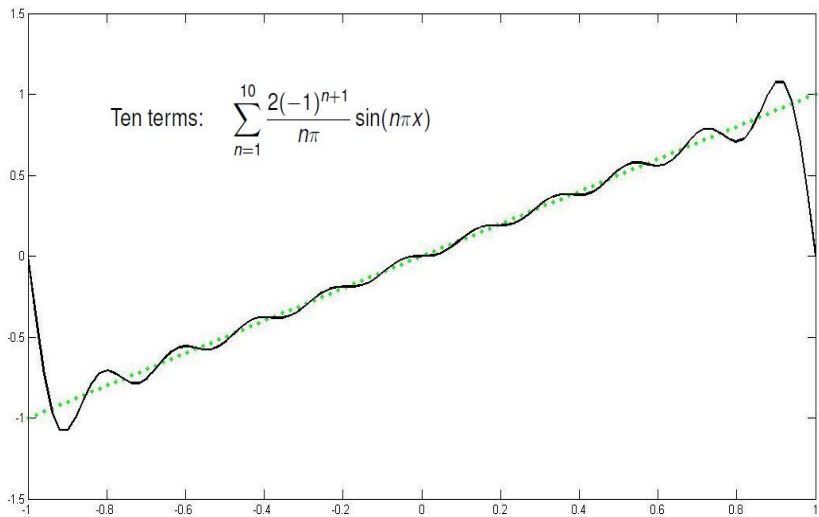


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with 10 terms of the Fourier series

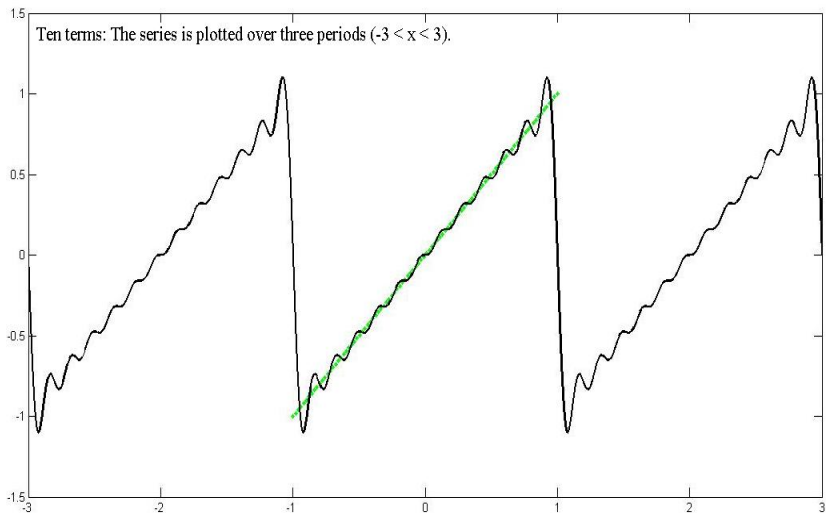


Figure: Plot of $f(x) = x$ for $-1 < x < 1$ with the Fourier series plotted on $(-3, 3)$. Note that the series repeats the profile every 2 units. At the jumps, the series converges to $(-1 + 1)/2 = 0$.

Section 18: Sine and Cosine Series

Functions with Symmetry

Recall some definitions:

Suppose f is defined on an interval containing x and $-x$.

If $f(-x) = f(x)$ for all x , then f is said to be **even**.

If $f(-x) = -f(x)$ for all x , then f is said to be **odd**.

For example, $f(x) = x^n$ is even if n is even and is odd if n is odd. The trigonometric function $g(x) = \cos x$ is even, and $h(x) = \sin x$ is odd.

Integrals on symmetric intervals

If f is an even function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 2 \int_0^p f(x) dx.$$

If f is an odd function on $(-p, p)$, then

$$\int_{-p}^p f(x) dx = 0.$$

Products of Even and Odd functions

$$\text{Even} \times \text{Even} = \text{Even},$$

and

$$\text{Odd} \times \text{Odd} = \text{Even}.$$

While

$$\text{Even} \times \text{Odd} = \text{Odd}.$$

So, suppose f **is even** on $(-p, p)$. This tells us that $f(x) \cos(nx)$ is **even** for all n and $f(x) \sin(nx)$ is **odd** for all n .

And, if f **is odd** on $(-p, p)$. This tells us that $f(x) \sin(nx)$ is **even** for all n and $f(x) \cos(nx)$ is **odd** for all n .

Fourier Series of an Even Function

If f is even on $(-p, p)$, then the Fourier series of f has only constant and cosine terms. Moreover

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

where

$$a_0 = \frac{2}{p} \int_0^p f(x) dx$$

and

$$a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

Fourier Series of an Odd Function

If f is odd on $(-p, p)$, then the Fourier series of f has only sine terms. Moreover

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

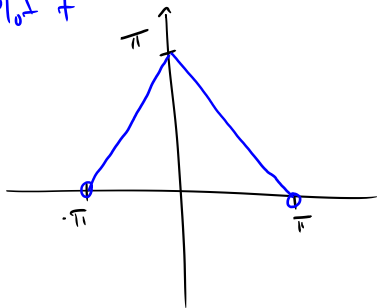
where

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

Find the Fourier series of f

$$f(x) = \begin{cases} x + \pi, & -\pi < x < 0 \\ \pi - x, & 0 \leq x < \pi \end{cases}$$

Plot f



f is even so $b_n = 0$ for all n

Using symmetry

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) dx = \frac{2}{\pi} \left[\pi x - \frac{x^2}{2} \right]_0^{\pi} = \frac{2}{\pi} \left[\pi^2 - \frac{\pi^2}{2} \right]$$

$$= \frac{2}{\pi} \left(\frac{\pi^2}{2} \right) = \pi$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi - x) \cos(nx) dx$$

$$= \frac{2}{\pi} \left[\frac{\pi - x}{n} \sin(nx) + \frac{1}{n^2} \cos(nx) \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{1}{n^2} \cos(n\pi) - \frac{1}{n^2} \cos(0) \right]$$

$$= \frac{2}{n^2\pi} \left((-1)^n - 1 \right)$$

$$a_0 = \pi \quad \text{and} \quad a_n = \frac{2}{n^2\pi} \left((-1)^n - 1 \right)$$

So

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi} \left((-1)^n - 1 \right) \cos(nx)$$

Half Range Sine and Half Range Cosine Series

Suppose f is only defined for $0 < x < p$. We can **extend** f to the left, to the interval $(-p, 0)$, as either an even function or as an odd function. Then we can express f with **two distinct** series:

$$\text{Half range cosine series } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{p}\right)$$

$$\text{where } a_0 = \frac{2}{p} \int_0^p f(x) dx \quad \text{and} \quad a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx.$$

$$\text{Half range sine series } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{p}\right)$$

$$\text{where } b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx.$$

Extending a Function to be Odd

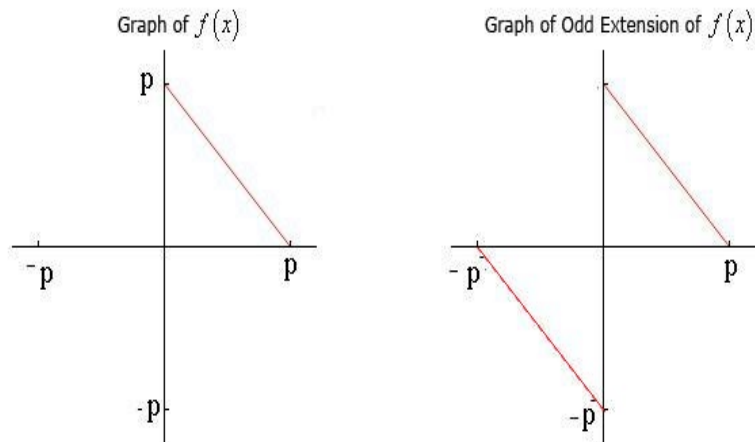


Figure: $f(x) = p - x$, $0 < x < p$ together with its **odd** extension.

Extending a Function to be Even

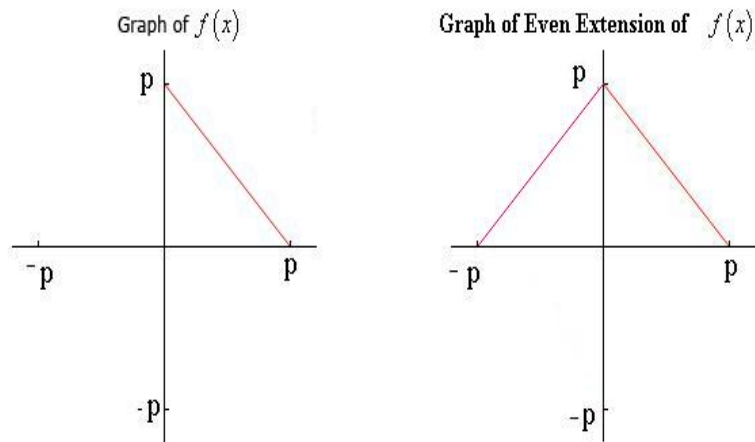


Figure: $f(x) = p - x$, $0 < x < p$ together with its **even** extension.